Computational Construction of New Projective Linear Codes of Dimension 4 over GF(11)

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ABSTRACT

This article investigates recent developments in computational research aimed at discovering new linear codes of dimension 4 over the Galois field $GF(11)$. We define an $[n,k,d]_q$ code as a k-dimensional subspace of $GF(q)^n$ with minimum Hamming distance d. Utilizing advanced computational techniques and algorithms, we study projective linear codes of dimension 4 derived from caps of degree 3, 4, and 5 containing an elliptic quadric. Our analysis focuses on linear codes within the length range $123 \leq n \leq 273$, representing the potential minimum Hamming distance for these codes across various n values. Through the computation of their weight enumerators, we introduce 969 new projective linear codes over $GF(11)$. Furthermore, we establish the following lower bounds for the maximum size of caps of degree r in $PG(3,11)$: $m_3(3,11) \geq 140$, $m_4(3,11) \geq 196$ and $m_5(3,11) \geq 273$. These results refine the known boundaries of coding theory parameters and enhance the reliability of error correction in high-order fields.

1. Introduction

Error-correcting codes are vital tools in digital communication, data storage, and cryptography [1, 2]. Linear codes over finite fields—particularly projective linear codes—provide powerful structures for improving data reliability by detecting and correcting transmission errors.

The study of linear codes over the Galois field $GF(q)$ has led to the discovery of numerous optimal or near-optimal codes. These codes are defined by fundamental parameters such as code length n, dimension k, and minimum Hamming distance d [7]. Recent research has furthered the development of computational boundaries for code performance [4, 5, 6, 8, 10, 14].

In projective geometry, caps—sets of points in a projective space $PG(k - 1, q)$ with no three points being collinear—are closely related to the construction of projective linear codes [3, 9]. The relationship between projective caps and linear codes allows researchers to explore geometric configurations that yield codes with large minimum distances. For instance, a $(k, 3)$-cap (often simply called a cap of degree 3) in $PG(3, q)$ with $q > 3$ satisfies the upper bound $k \leq 2q^2 + 1 - \alpha(q)$ [12]. For a comprehensive overview of linear codes, one may refer to [11].

This paper focuses on the computational construction of new projective linear codes of dimension 4 over the Galois field $GF(11)$. By employing computer-assisted search methods and the algorithmic enumeration of projective caps of degrees 3, 4, and 5 containing an elliptic quadric, we have identified and characterized 969 new codes within the length range $123 \leq n \leq 273$.

2. Preliminaries

We begin by recalling fundamental definitions and theorems from coding theory and projective geometry [2, 4].

Definition 2.1 (Linear Code) [2].

A linear code C of length n and dimension k over a finite field $GF(q)$ is a k-dimensional subspace of $GF(q)^n$. The minimum Hamming distance d of C is the minimum number of non-zero components in any non-zero codeword of C. Such a code is denoted by $[n, k, d]_q$.

Definition 2.2 (Weight Enumerator) [2].

Let C be a linear code of length n over $GF(q)$. The weight enumerator of C is a polynomial $W_C(x)$ that categorizes all codewords based on their Hamming weight (the number of non-zero entries). It is formally defined as:

$$W_C(x) = \sum_{i=0}^{n} A_i x^i \tag{1}$$

where A_i is the number of codewords in C with weight exactly i. Note that for a code of dimension $k = 4$, the sum of all coefficients $W_C(1)$ must be equal to q^4.

Definition 2.3 (Projective Linear Code) [3].

A **projective linear code** is a linear code whose generator matrix columns correspond to points of a projective space $PG(k-1,q)$ such that no two columns are scalar multiples of one another.

Definition 2.4 (wn-equivalence).

Two projective linear codes C_1 and C_2 are said to be wn-equivalent if $W_{C_1}(x) = W_{C_2}(x)$. Otherwise, they are wn-inequivalent.

Theorem 2.1 [11].

Let S be a cap in $PG(k-1,q)$ of size n. The code generated by the homogeneous coordinates of the points in S is a projective linear $[n,k,d]_q$ code. The minimum distance d satisfies:

$$d = n - \max_H |S \cap H|,$$

where H ranges over all hyperplanes of $PG(k-1,q)$.

Theorem 2.2 (Elliptic Quadric Cap Construction) [3, 4].

If S is a cap containing an elliptic quadric in $PG(3, 11)$, then the associated linear code C_S is projective and achieves a minimum distance greater than or equal to the order of the elliptic quadric.

Definition 2.5 ((v, κ)-cap in $PG(n, q)$).

A (v, κ)-cap in the projective space $PG(n, q)$ is a set of v points such that at most κ points lie on the same line, but no $\kappa + 1$ points are collinear. Such caps are called caps of degree κ. The quantity $m_\kappa(n, q)$ denotes the maximum cardinality v for which a (v, κ)-cap exists in $PG(n, q)$ [12].

In this article, we classify projective linear codes up to wn-equivalence. By doing so, we establish new lower bounds for $m_\kappa(n, q)$ for $\kappa \in \{3,4,5\}$. Furthermore, we identify 969 new linear codes that are not currently documented in the Grassl database [13]. Most existing bounds and constructions for $[n, k, d]_q$ codes in [13] focus on $GF(q)$ for $q \in \{2,3,4,5,7,8,9\}$, whereas this work provides results for $GF(11)$.

3. Computational Method and Results

We implemented a combination of exhaustive and heuristic search algorithms to enumerate all non-wn-equivalent projective linear codes within $PG(3, 11)$. Each configuration was analyzed to verify its projectivity and to calculate its minimum distance using computational algebra systems, specifically MATLAB and Fortran PowerStation.

Consider an elliptic quadric form over the finite field $GF(11)$ defined by:

$$Q(x_1, x_2, x_3, x_4) = \sum_{1 \le i,j \le 4} q_{i,j} x_i x_j.$$

Let the non-zero coefficients be $q_{2,2} = 4$, $q_{3,3} = 1$, and $q_{1,4} = 6$, with all other $q_{i,j} = 0$. The quadratic form reduces to:

$Q(x_1, x_2, x_3, x_4) = 4x_2^2 + x_3^2 + 6x_1x_4$. This quadratic form can be represented in matrix form as $X^T M X$, where $X = (x_1, x_2, x_3, x_4)^T$, and M is the symmetric matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}.$$

In $GF(11)$, the term $6x_1x_4$ is represented in the symmetric matrix M by splitting the coefficient $q_{1,4} = 6$ into two entries: $m_{1,4}$ and $m_{4,1}$. Since the matrix form X^TMX results in $2m_{1,4}m_{4,1}$, we calculate the entry as $6 \cdot 2^{-1} \ (mod\ 11)$.

The modular inverse of 2 is 6 (since $2 \cdot 6 = 12 \equiv 1 \ (mod\ 11)$). Therefore: $6 \times 6 \equiv 3 \ (mod\ 11)$.

Thus, $m_{1,4} = m_{4,1} = 3$.

The determinant of M is non-zero in $GF(11)$, confirming that Q defines a non-degenerate quadratic form. Specifically, this defines an elliptic quadric in $PG(3, 11)$, often denoted as $Q(3,11)$ [3].

3.1 Construction of a Projective Linear Code from an Elliptic Quadric

To construct optimal projective linear codes, we utilize the geometric properties of caps and quadrics in projective space. The primary objective is to iteratively expand a known cap to find new projective linear codes with improved parameters.

The Construction Algorithm:

Let G_n be the generator matrix of a projective $[n, k, d]_q$ code, where the columns of G_n correspond to a (n, κ)-cap S in $PG(k-1, q)$. To find a new code of length $n + m$, we follow these steps:

1. **Selection of the Base Set:** Begin with an initial set of points $S \subset PG(k-1, q)$ (such as an elliptic quadric $Q(3,11)$ where $n = q^2 + 1$).
2. **Search Space Definition:** Define the candidate set of points $P_{cand} = PG(k-1, q) \backslash S$.
3. **Iterative Augmentation:** Select a subset of points $\{P_1, P_2, \cdots, P_m\} \subset P_{cand}$ to append to the existing set S. The selection is governed by the constraint that the new set $S' = S \cup \{P_1, P_2, \cdots, P_m\}$ must not exceed the desired cap degree κ.

4. **Matrix Evolution:** Construct the new generator matrix $G_{n+m} = [G_{n+m}|\, P_1 \;\; P_2 \cdots P_m]$. This matrix corresponds to a projective linear code of length $n+m$.
5. **Parameter Verification:** Calculate the **minimum distance** d' of the new code using $d' = (n+m) - \max_{H}|S' \cap H|$.
   - Compute the **weight enumerator** $W_{C'}(x)$.
   - Compare the resulting $[n+m, k, d']_q$ code against existing databases (e.g., the Grassl database) to determine if the code is new or optimal.

Following this generalized approach, we started with the generator matrix G_{122} of the elliptic quadric $Q(3,11)$. By appending columns from $PG(3,11)\backslash Q(3,11)$, we constructed a sequence of expanding codes.

For brevity, the point (a,b,c,d) in $PG(3,11)$ will be represented by the string $abcd$. Also, we represent the generator matrix as an ordered set of columns. Let $G[n,d]$ be the generator matrix of a projective linear code $C(n,d)$ with parameters $[n,4,d]_{11}$.

For example, adding the column 1112 to G_{122} yields:

$$G[123,4] = G_{122} \cup \{(1112)\}$$

This generates a code C_1 with parameters $[123,4,110]_{11}$ and weight enumerator:

$$W_{C_1}(x) = 1 + 1210x^{110} + 12210x^{111} + 120x^{121} + 1100x^{122}.$$

By continuing this iterative addition, ensuring no four points are collinear to maintain a cap of degree $\kappa = 3$, we identified further codes:

$$G(124,110) = G_{122} \cup \{0190, 1112\},$$
$$G(125,110) = G_{122} \cup \{160X, 0190, 1112\},$$

where X denotes the element $10 \in GF(11)$.

The weight enumerator for $C(124,110)$ is given by:

$$W_{C(124,110)}(x) = 1 + 110x^{110} + 2200x^{111} + 11110x^{112} + 10x^{121} + 220x^{122} + 990x^{123}.$$

The projective linear code $C(125,110)$ generated by $G(125,110)$ has the weight enumerator:

$$W_{C(125,110)}(x) = 1 + 10x^{110} + 300x^{111} + 3000x^{112} + 10110x^{113} + 30x^{122} + 300x^{123} + 890x^{124}.$$

By continuing this iterative augmentation, we identified a total of 40 new projective linear codes. These results are summarized in Table 1 and lead to the following theorem regarding the lower bound of $(n, 3)$-caps.

Theorem 3.1. In the projective space $PG(3, 11)$, there exist at least 40 new projective linear codes of dimension $k = 4$. Furthermore, the maximum cardinality of a cap of degree 3 in $PG(3, 11)$ satisfies:

$$m_3(3,11) \geq 140.$$

Proof.

The new projective linear codes, whose weight distributions are provided in Table 1, correspond to $(n, 3)$-caps in $PG(3, 11)$. The generator matrices for the codes $C(140, 122)$ and $C(140, 123)$ are constructed by appending 18 points to the elliptic quadric $Q(3,11)$.

Specifically:

$G(140, 122) = G_{122}$ ∪ {1486, 1351, 1953, 1222, 1X3X, 11X2, 1568, 1411, 1142, 1615, 15X8,
0155, 1476, 0189, 1165, 1091, 0190, 1112}.
$G(140, 123) = G_{122}$ ∪ {1108, 0196, 0186, 1458, 0125, 1570, 1671, 1628, 0191, 0114, 0155, 0133, 1091, 1939, 1697, 0174, 0019, 1112}.

Let $A(140, 122)$ and $A(140, 123)$ denote the sets of points corresponding to these generator matrices. Computational verification using exhaustive search confirms that no additional point $P \in PG(3, 11)$ can be added to either $A(140, 122)$ or $A(140, 123)$ without violating the degree 3 constraint (i.e., without creating a line with 4 points).

Therefore, the point sets $A(140, 122)$ and $A(140, 123)$ form complete $(\kappa, 3)$-caps of size $\kappa = 140$, which establishes the lower bound $m_3(3,11) \geq 140$.

Table1: New $[n, 4, d]$ projective linear codes with weight enumerator over $GF(11)$

Code ID	**Weight enumerator**	***n***	***d***	**Generation Time**
$C(124{,}110)$	$W_{C(124,110)}(x) = 1 + 110x^{110} + 2200x^{111} + 11110x^{112} + 10x^{121} + 220x^{122} + 990x^{123}$	124	110	2 second
$C(125{,}110)$	$W_{C(125,110)}(x) = 1 + 10x^{110} + 300x^{111} + 3000x^{112} + 10110x^{113} + 30x^{122} + 300x^{123} + 890x^{124}$	125	110	2 second
$C(126{,}110)$	$W_{C(126,110)}(x) = 1 + 10x^{110} + 90x^{111} + 280x^{112} + 3970x^{113} + 9070x^{114} + 20x^{122} + 50x^{123} + 320x^{124} + 830x^{125}$	126	110	2 second
$C(126{,}111)$	$W_{C(126,111)}(x) = 1 + 40x^{111} + 540x^{112} + 3650x^{113} + 9200x^{114} + 60x^{123} + 360x^{124} + 800x^{125}$	126	111	5 second
$C(127{,}111)$	$W_{C(127,111)}(x) = 1 + 10x^{111} + 120x^{112} + 560x^{113} + 4470x^{114} + 8230x^{115} + 10x^{122} + 10x^{123} + 100x^{124} + 330x^{125} + 770x^{126}$	127	111	5 second
$C(127{,}112)$	$W_{C(127,112)}(x) = 1 + 100x^{112} + 780x^{113} + 4190x^{114} + 8350x^{115} + 120x^{124} + 360x^{125} + 740x^{126}$	127	112	7 second
$C(128{,}112)$	$W_{C(128,112)}(x) = 1 + 10x^{112} + 160x^{113} + 1060x^{114} + 4620x^{115} + 7570x^{116} + 200x^{125} + 320x^{126} + 700x^{127}$	128	112	7 second
$C(128{,}113)$	$W_{C(128,113)}(x) = 1 + 170x^{113} + 1080x^{114} + 4590x^{115} + 7580x^{116} + 30x^{124} + 120x^{125} + 390x^{126} + 680x^{127}$	128	113	8 second
$C(129{,}112)$	$W_{C(129,112)}(x) = 1 + 10x^{112} + 250x^{114} + 1400x^{115} + 4870x^{116} + 6890x^{117} + 270x^{126} + 300x^{127} + 650x^{128}$	129	112	9 second
$C(129{,}113)$	$W_{C(129,113)}(x) = 1 + 10x^{113} + 270x^{114} + 1370x^{115} + 4880x^{116} + 6890x^{117} + 40x^{125} + 160x^{126} + 400x^{127} + 620x^{128}$	129	113	9 second
$C(129{,}114)$	$W_{C(129,114)}(x) = 1 + 190x^{114} + 1530x^{115} + 4840x^{116} + 6860x^{117} + 30x^{124} + 40x^{125} + 120x^{126} + 360x^{127} + 670x^{128}$	129	114	9 second
$C(130{,}113)$	$W_{C(130,113)}(x) = 1 + 10x^{113} + 10x^{114} + 380x^{115} + 1680x^{116} + 5090x^{117} + 6250x^{118} + 40x^{126} + 260x^{127} + 320x^{128} + 600x^{129}$	130	113	10 second
$C(130{,}114)$	$W_{C(130,114)}(x) = 1 + 50x^{114} + 320x^{115} + 1710x^{116} + 5100x^{117} + 6240x^{118} + 40x^{126} + 270x^{127} + 300x^{128} + 610x^{129}$	130	114	11 second
$C(130{,}115)$	$W_{C(130,115)}(x) = 1 + 310x^{115} + 1880x^{116} + 4990x^{117} + 6240x^{118} + 10x^{124} + 20x^{125} + 70x^{126} + 120x^{127} + 380x^{128} + 620x^{129}$	130	115	11 second

Code ID	Weight enumerator	n	d	Generation Time
$C(131,114)$	$W_{C(131,114)}(x) = 1 + 10x^{114} + 30x^{115} + 440x^{116} + 2040x^{117} + 5320x^{118} + 5580x^{119} + 10x^{123} + 30x^{126} + 40x^{127} + 170x^{128} + 430x^{129} + 540x^{130}$	131	114	12 second
$C(131,115)$	$W_{C(131,115)}(x) = 1 + 60x^{115} + 480x^{116} + 1820x^{117} + 5570x^{118} + 5490x^{119} + 10x^{123} + 10x^{125} + 20x^{126} + 30x^{127} + 180x^{128} + 430x^{129} + 540x^{130}$	131	115	12 second
$C(132,115)$	$W_{C(132,115)}(x) = 1 + 10x^{115} + 110x^{116} + 520x^{117} + 2360x^{118} + 5330x^{119} + 5090x^{120} + 10x^{125} + 20x^{127} + 80x^{128} + 210x^{129} + 400x^{130} + 500x^{131}$	132	115	15 second
$C(132,116)$	$W_{C(132,116)}(x) = 1 + 110x^{116} + 540x^{117} + 2400x^{118} + 5240x^{119} + 5130x^{120} + 10x^{126} + 40x^{127} + 70x^{128} + 170x^{129} + 440x^{130} + 490x^{131}$	132	116	15 second
$C(133,115)$	$W_{C(133,115)}(x) = 1 + 10x^{115} + 10x^{116} + 120x^{117} + 780x^{118} + 2320x^{119} + 5740x^{120} + 4440x^{121} + 10x^{125} + 10x^{127} + 20x^{128} + 110x^{129} + 210x^{130} + 370x^{131} + 490x^{132}$	133	115	15 second
$C(133,116)$	$W_{C(133,116)}(x) = 1 + 20x^{116} + 100x^{117} + 780x^{118} + 2490x^{119} + 5490x^{120} + 4540x^{121} + 20x^{126} + 20x^{120} + 120x^{129} + 190x^{130} + 380x^{131} + 490x^{132}$	133	116	17 second
$C(133,117)$	$W_{C(133,117)}(x) = 1 + 70x^{117} + 740x^{118} + 2870x^{119} + 5070x^{120} + 4670x^{121} + 10x^{126} + 40x^{127} + 10x^{128} + 100x^{129} + 180x^{130} + 360x^{131} + 520x^{132}$	133	117	17 second
$C(134,116)$	$W_{C(134,116)}(x) = 1 + 10x^{116} + 70x^{117} + 140x^{118} + 610x^{119} + 3330x^{120} + 5050x^{121} + 4200x^{122} + 20x^{128} + 30x^{129} + 140x^{130} + 270x^{121} + 260x^{132} + 500x^{133}$	134	116	20 second
$C(134,117)$	$W_{C(134,117)}(x) = 1 + 30x^{117} + 140x^{118} + 960x^{119} + 2910x^{120} + 5110x^{121} + 4270x^{122} + 10x^{128} + 50x^{129} + 80x^{130} + 350x^{131} + 250x^{132} + 480x^{133}$	134	117	20 second
$C(134,118)$	$W_{C(134,118)}(x) = 1 + 90x^{118} + 930x^{119} + 3270x^{120} + 4830x^{121} + 4300x^{122} + 10x^{127} + 40x^{128} + 50x^{129} + 110x^{130} + 140x^{131} + 370x^{132} + 500x^{133}$	134	118	20 second
$C(135,117)$	$W_{C(135,117)}(x) = 1 + 10x^{117} + 20x^{118} + 230x^{119} + 1030x^{120} + 3140x^{121} + 5280x^{122} + 3710x^{123} + 10x^{126} + 10x^{128} + 10x^{129} + 50x^{130} + 110x^{131} + 240x^{132} + 360x^{133} + 430x^{134}$	135	117	24 second
$C(135,118)$	$W_{C(135,118)}(x) = 1 + 10x^{118} + 150x^{119} + 1110x^{120} + 3480x^{121} + 4790x^{122} + 3880x^{123} +$	135	118	24 second

Code ID	Weight enumerator	n	d	Generation Time
	$10x^{127} + 40x^{129} + 80x^{130} + 90x^{131} + 190x^{132} + 320x^{133} + 490x^{134}$			
$C(135{,}119)$	$W_{C(135,119)}(x) = 1 + 150x^{119} + 1160x^{120} + 3450x^{121} + 4750x^{122} + 3910x^{123} + 20x^{128} + 30x^{129} + 70x^{130} + 120x^{131} + 150x^{132} + 350x^{133} + 480x^{134}$	135	119	24 second
$C(136{,}118)$	$W_{C(136,118)}(x) = 1 + 10x^{118} + 40x^{119} + 210x^{120} + 1460x^{121} + 3070x^{122} + 5320x^{123} + 3310x^{124} + 10x^{125} + 10x^{129} + 40x^{130} + 10x^{131} + 110x^{132} + 270x^{133} + 410x^{134} + 360x^{135}$	136	118	29 second
$C(136{,}119)$	$W_{C(136,119)}(x) = 1 + 10x^{119} + 240x^{120} + 1330x^{121} + 3590x^{122} + 4760x^{123} + 3490x^{124} + 10x^{127} + 10x^{129} + 40x^{130} + 80x^{131} + 100x^{132} + 190x^{133} + 340x^{134} + 450x^{135}$	136	119	29 second
$C(136{,}120)$	$W_{C(136,120)}(x) = 1 + 280x^{120} + 1370x^{121} + 3330x^{122} + 5050x^{123} + 3390x^{124} + 10x^{127} + 10x^{128} + 20x^{130} + 80x^{131} + 160x^{132} + 150x^{133} + 330x^{134} + 460x^{135}$	136	120	29 second
$C(137{,}119)$	$W_{C(137,119)}(x) = 1 + 10x^{119} + 40x^{120} + 280x^{121} + 1720x^{122} + 3280x^{123} + 5050x^{124} + 3050x^{125} + 10x^{120} + 50x^{131} + 20x^{132} + 110x^{133} + 280x^{134} + 410x^{135} + 330x^{136}$	137	119	31 second
$C(137{,}120)$	$W_{C(137,120)}(x) = 1 + 10x^{120} + 340x^{121} + 1580x^{122} + 3640x^{123} + 4720x^{124} + 3130x^{125} + 10x^{127} + 40x^{130} + 10x^{131} + 80x^{132} + 110x^{133} + 210x^{134} + 350x^{135} + 410x^{136}$	137	119	31 second
$C(137{,}121)$	$W_{C(137,121)}(x) = 1 + 290x^{121} + 1930x^{122} + 3130x^{123} + 4960x^{124} + 3120x^{125} + 80x^{132} + 150x^{133} + 240x^{134} + 420x^{135} + 320x^{136}$	137	121	31 second
$C(138{,}120)$	$W_{C(138,120)}(x) = 1 + 20x^{120} + 70x^{121} + 490x^{122} + 1590x^{123} + 3570x^{124} + 5020x^{125} + 2660x^{126} + 10x^{129} + 30x^{130} + 30x^{132} + 60x^{133} + 120x^{134} + 190x^{135} + 500x^{136} + 280x^{137}$	138	120	33 second
$C(138{,}121)$	$W_{C(138,121)}(x) = 1 + 40x^{121} + 520x^{122} + 1670x^{123} + 3690x^{124} + 4690x^{125} + 2810x^{126} + 10x^{128} + 30x^{130} + 10x^{131} + 20x^{132} + 40x^{133} + 100x^{134} + 300x^{135} + 400x^{136} + 310x^{137}$	138	121	33 second

Code ID	Weight enumerator	n	d	Generation Time
$C(139{,}121)$	$W_{C(139,121)}(x) = 1 + 20x^{121} + 100x^{122} + 570x^{123} + 1810x^{124} + 3730x^{125} + 4780x^{126} + 2410x^{127} + 20x^{130} + 20x^{131} + 60x^{133} + 50x^{134} + 120x^{135} + 200x^{136} + 480x^{137} + 270x^{138}$	139	121	37 second
$C(139{,}122)$	$W_{C(139,122)}(x) = 1 + 40x^{122} + 660x^{123} + 1890x^{124} + 3790x^{125} + 4480x^{126} + 2560x^{127} + 10x^{128} + 10x^{130} + 20x^{131} + 20x^{132} + 10x^{133} + 60x^{134} + 100x^{135} + 290x^{136} + 430x^{137} + 270x^{138}$	139	122	37 second
$C(139{,}123)$	$W_{C(139,123)}(x) = 1 + 460x^{123} + 2650x^{124} + 3110x^{125} + 4600x^{126} + 2610x^{127} + 110x^{134} + 220x^{135} + 190x^{136} + 410x^{137} + 280x^{138}$	139	123	37 second
$C(140{,}122)$	$W_{C(140,122)}(x) = 1 + 20x^{122} + 120x^{123} + 710x^{124} + 2000x^{125} + 3830x^{126} + 4560x^{127} + 2180x^{128} + 40x^{131} + 10x^{133} + 60x^{134} + 50x^{135} + 130x^{136} + 210x^{137} + 470x^{138} + 250x^{139}$	140	122	41 second
$C(140{,}123)$	$W_{C(140,123)}(x) = 1 + 60x^{123} + 610x^{124} + 2390x^{125} + 3990x^{126} + 3890x^{127} + 2480x^{128} + 20x^{130} + 10x^{132} + 20x^{133} + 70x^{134} + 60x^{135} + 130x^{136} + 260x^{137} + 290x^{138} + 360x^{139}$	140	123	41 second

Theorem 3.2. In the projective space $PG(3, 11)$, there exist 929 new projective linear codes of dimension $k = 4$. These are produced by extending the generator matrix of the $[140, 4, d]_{11}$ code. Furthermore, the maximum cardinality for caps of degree 4 and 5 satisfies:

$$m_4(3{,}11) \geq 196 \text{ and } m_5(3{,}11) \geq 273.$$

Proof.

Recall from Theorem 3.1 that the generator matrix G_{140} corresponds to the (140, 3)-cap, $A(140, 123)$. By iteratively adding points from $PG(3, 11) \setminus A(140, 123)$, we obtained several new code families.

(1). Extension to Length $n = 141$

By adding a single point, we obtained the following matrices:

$$G(141,123) = G_{140} \cup \{161X\},$$
$$G(141,124) = G_{140} \cup \{1794\}.$$

The corresponding weight enumerators are:

$W_{C(141,123)}(x) = 1 + 10x^{123} + 110x^{124} + 770x^{125} + 2530x^{126} + 3920x^{127} + 3870x^{128} + 2210x^{129} + 20x^{131} + 10x^{133} + 40x^{134} + 50x^{135} + 80x^{136} + 120x^{137} + 270x^{138} + 320x^{139} + 310x^{140}$,

$W_{C(141,124)}(x) = 1 + 130x^{124} + 790x^{125} + 2490x^{126} + 4010x^{127} + 3690x^{128} + 2310x^{129} + 20x^{131} + 10x^{133} + 20x^{134} + 70x^{135} + 60x^{136} + 140x^{137} + 290x^{138} + 320x^{139} + 290x^{140}$.

(2). Extension to Length $n = 142$

Adding pairs of points yielded:

$$G(142,123) = G_{140} \cup \{161X, 1366,\},$$
$$G(142,124) = G_{140} \cup \{161X, 1960,\},$$
$$G(142,125) = G_{140} \cup \{1794, 1725\}.$$

(3). Caps of Degree 4 ($m_4(3,11) \geq 196$)

Continuing this extension, we identified 311 projective linear codes representing caps of degree 4. Specifically, we found a complete (196, 4)-cap, $A(196,171)$, by adding 56 points to G_{140}. The additional points include:

{1000, 19X1, 1495, 0179, 1899, 129X, 1460, 1577, 1348, 1651, 1X18, 1237, 1894, 0140, 0014, 1806, 1464, 1XX5, 18X6, 01X8, 1662, 1045, 0142, 1X71, 1248, 1229, 1301, 1710, 1946, 1181, 19X5, 1307, 1760, 0176, 161X, 1484, 1748, 1441, 1367, 1668, 1952, 1718, 170X, 17X7, 0016, 1033, 1747, 124X, 1X65, 1274, 1214, 1859, 1853, 1366, 115X, 1347}

Exhaustive computational search confirms that $A(196,171)$ is complete (196, 4)-cap, as no further points can be added without creating a line with 5 points.

As results for caps of degree 4, by iteratively extending the previously established generator matrices, we obtained 311 new projective linear codes of dimension $k = 4$ with varying minimum distances. The relationship between the code length n and the possible minimum distances d for these configurations is illustrated in Figures 1 and 2.

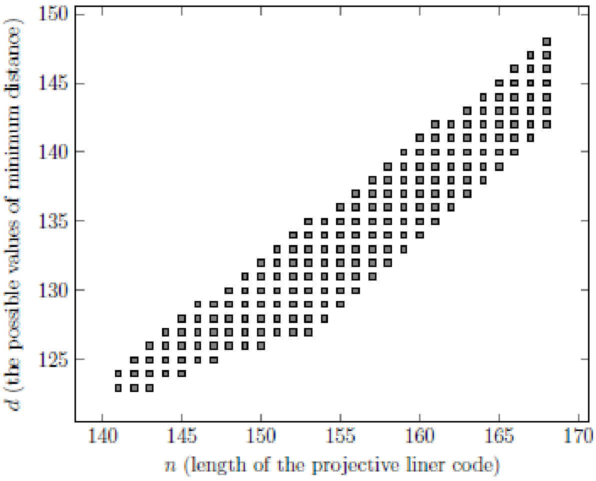

Figure 1: The possible values of $d_{11}(n, 4)$ for $n = 141, 142, \ldots, 168$.

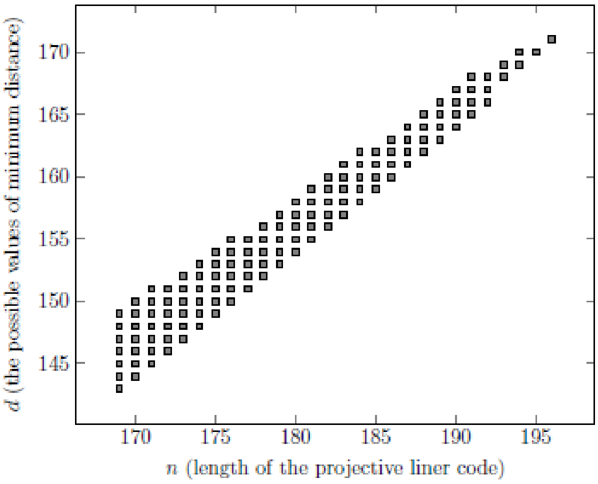

Figure 2: The possible values of d_{11}(n, 4) for n = 169, 170, . . . , 196

The columns of the generator matrix for each of these codes correspond to the coordinates of a cap of degree 4 in the projective space $PG(3, 11)$. A cap of degree 4 ensures that no five points are collinear, allowing for larger code lengths while maintaining a high minimum distance.

Specifically, we identified a significant configuration representing a (196, 4)-cap. This cap corresponds to a projective linear code with length $n = 196$ and minimum distance $d = 171$.

(4). Caps of Degree 5 ($m_5(3,11) \geq 273$)

By extending $A(196, 171)$, we discovered 618 additional projective linear codes corresponding to caps of degree 5. We identified a complete (273, 5)-cap, $A(273, 239)$, by adding 77 points to G_{196}. Key additional points include:

{0010, 1XXX, 13XX, 1374, 1485, 1560, 0156, 1316, 1808, 1564, 1597, 1128, 1358, 1930, 1146, 1887, 1304, 1X02, 12X5, 1987, 1571, 1205, 1589, 1269, 1064, 10X5, 0123, 1400, 16X9, 1511, 1191, 198X, 1676, 1066, 1941, 1516, 1X31, 1150, 0115, 0130, 1114, 1514, 1903, 1652, 18X0, 1136, 0188, 1933, 010X, 110X, 1750, 1288, 16X8, 1009, 1X8X, 1351, 1010, 1XX3, 1713, 0192, 1638, 1013, 0172, 0183, 1X42, 1375, 1529, 1364, 0128, 1281, 1630, 1840, 1353, 0153, 1677, 10X6, 1884}.

Computational verification confirms that $A(273, 239)$ is complete, implying $m_5(3,11) \geq 273$.

As results for caps of degree 5, by further extending the generator matrix of the projective linear code corresponding to the (196, 4)-cap, we obtained 618 new projective linear codes of dimension $k = 4$. These codes exhibit a variety of minimum distances, the distributions of which are illustrated in Figures 3 and 4.

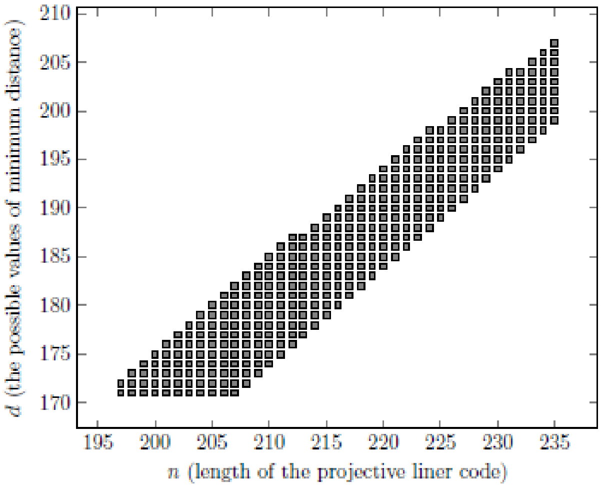

Figure 3: The possible values of d_{11}(n, 4) for n = 197, 198, . . . , 235.

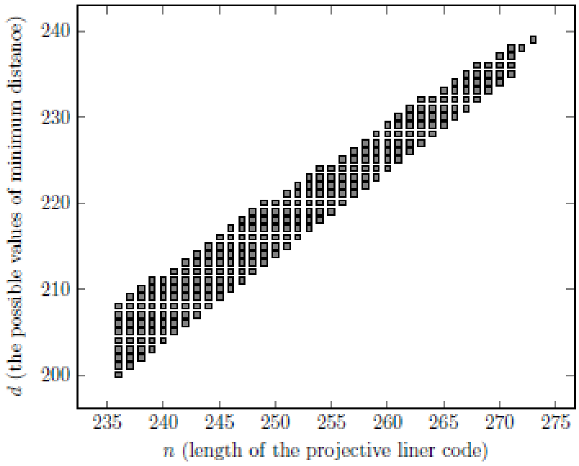

Figure 4: The possible values of d_{11}(n, 4) for n = 236, 237, . . . , 273

The columns of the generator matrices for these codes correspond to the points of caps of degree 5 in $PG(3, 11)$. A cap of degree 5 is defined such that no six points are collinear. Within this set of results, we identified a significant configuration representing a (273, 5)-cap. This cap corresponds to a projective linear code with length $n = 273$ and a minimum distance of $d = 239$.

4. Conclusions

In this paper, we have explored the constructive relationship between projective geometry and coding theory within the projective space $PG(3, 11)$. By utilizing the elliptic quadric $Q(3,11)$ as a foundational $(122, 2)$-cap, we successfully implemented an iterative computational algorithm to expand this structure into higher-degree caps.

Our research led to the following key contributions:

1. Discovery of New Codes: We identified and characterized 969 new projective linear codes of dimension $k = 4$ over $GF(11)$. These codes are currently undocumented in the Grassl database, filling a significant gap in the literature for codes over larger prime fields.
2. Classification by wn-equivalence: Through the analysis of weight enumerators, we classified these codes into non-equivalent classes, providing a clear map of their performance and structural properties.
3. New Lower Bounds: We established lower bounds for the maximum cardinality of caps of degrees 3, 4, and 5. Specifically, we proved:

4. $m_3(3,11) \geq 140$
5. $m_4(3,11) \geq 196$
6. $m_5(3,11) \geq 273$
7. Completeness Verification: By demonstrating the existence of complete caps for each degree, we have provided benchmark configurations that future research can use to test further optimality.
8. The results confirm that the algorithmic enumeration of caps containing an elliptic quadric is an effective strategy for producing projective linear codes with high minimum distances. Future work could extend this methodology to higher dimensions $PG(n,q)$ or explore the properties of these codes in the context of advanced cryptographic protocols.

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بناء حسابي لشفرات خطية إسقاطيه جديدة
ذات بُعد 4 على حقل كالوا
GF (11)

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المستخلص

تتناول هذه المقالة التطورات الحديثة في البحوث الحسابية التي تهدف إلى اكتشاف شفرات خطية جديدة ذات بُعد 4 على حقل غالوا $GF(11)$. نُعرّف الشفرة الخطية $[n,k,d]_q$ على أنه فضاء جزئي ذو بُعد k من $GF(q)^n$ ذو مسافة هامينغ دنيا d. باستخدام تقنيات وخوارزميات حسابية متقدمة، ندرس الشفرات الخطية الإسقاطية ذات البُعد 4 المُشتقة من أغطية من الدرجة 3 و4 و5 وتحتوي على سطح تربيعي إهليلجي. يركز تحليلنا على الشفرات الخطية ضمن نطاق الطول $123 \leq n \leq 273$، والذي يُمثل مسافة هامينغ الدنيا المُحتملة لهذه الشفرات عبر قيم n المختلفة. من خلال حساب مُعدِّدات أوزانها، نُقدم 969 شفرة خطيَّة إسقاطيَّة جديدَّة على $GF(11)$. علاوة على ذلك، نحدد الحدود الدنيا التالية لأقصى حجم للأغطية من الدرجة r في $PG(3,11)$: $m_3(3,11) \geq 140$ و $m_4(3,11) \geq 196$ و $m_5(3,11) \geq 273$. تُحسّن هذه النتائج الحدود المعروفة لمعاملات نظرية الترميز، وتُعزز موثوقية تصحيح الأخطاء في الحقول ذات الرتب العليا.

الكلمات المفتاحية: الشفرات الخطية الإسقاطية، الفضاء الإسقاطي، الحقول المنتهية، معدِّد الأوزان.