Solving Frobenius Equation in the Large Mathieu Groups 1

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Abstract

In this paper, the solutions of Frobenius equation $x^n = a$ are investigated in the large Mathieu groups M_{22} and M_{23} where n is a positive integer, and x, a are conjugacy classes. Furthermore, three algorithms in GAP system are provided to compute the number of solutions and solutions itself (if exists) for this equation.

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1. Introduction:

If G is a finite group, a fundamental result of Frobenius states that the number of elements, which satisfy the equation $x^n = e$, where e is the identity of G, n divides the order of G, is divisible by n. If x conjugate to $y \in G$ and $x^n = a$, then $y^n = b$, where b is a conjugate to a, this means Frobenius equation is a class equation (on conjugacy classes of G). Let $A_n(G)=\{x \in G: x^n = a\}$ the set of solutions in G to the equation $x^n = a$, the Frobenius equation have been studied by many authors such as, Chowla, Herstein and Scott [1]. Solomon, L. [2]. Lam, T.Y [3]. Yoshida, T. [4]. Taban. S.A [5]. Amit, A. and Vishne, U.(2011) [6]. Kholil, S.M. [7]. Shamkhi, R.H. [8], and many others.

2. Mathieu Groups:

The classification of all finite simple began in 1891, with the discovery of two of the sporadic simple groups by Franch Mathematician E′mile Mathieu [9]. In 1861 and 1873, Emile Mathieu published two papers revealing the first five sporadic simple groups, aptly named the Mathieu [9].The Mathieu groups M_i , i∈{11,12,22,23,24} are divided into the small Mathieu groups M_{11} and M_{12} and the large Mathieu groups M_{22} , M_{23} and M_{24}.For the conjugacy classes for these groups, we are using algorithm program GAP 4.10.2 www.gap-system.org [10]. For more information about Mathieu groups see Ivanov, A. A. (2018) [11].

3. The Mathieu Group M_{22} [9, 12]:

M_{22} is one of the large Mathieu groups, a sporadic simple group of order $443520=2^7.3^2.5.7.11$ can be given as the subgroup of S_{22} generated by the permutations.

G=(1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22)

H=(1,4,5,9,3)(2,8,10,7,6)(12,15,16,20,14)(13,19,21,18,17)

I=(11,22)(1,21)(2,10,8,6)(12,14,16,20)(4,17,3,13)(5,19,9,18)

M_{22}=< G, H, I >.

3.1 Conjugacy Classes of M_{22}:

In this section, we construct the following algorithm by using GAP program in order to find the conjugacy classes of M_{22} which is a subgroup from the symmetric group S_{22}.

Algorithm 1:

```
gap>s22:=Group((1,2),(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22));
Group([(1,2),(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)])
gap>M22:=Group((1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22),(1,4,5,9,3)
(2,8,10,7,6)(12,15,16,20,14)(13,19,21,18,17),(11,22)(1,21)(2,10,8,6)(12,14,16,20)(4,17,3,13)(5,1
9,9,18));
Group([(1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22),(1,4,5,9,3)(2,8,10,7,6)
(12,15,16,20,14)(13,19,21,18,17),(1,21)(2,10,8,6)(3,13,4,17)(5,19,9,18)(11,22)   (12,14,16,20)])
gap>ccl:=ConjugacyClasses(M22);
[()^G,(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)^G,
(1,11,3,10,6,4,15)(2,21,17,22,16,12,9)(5,13,18,7,14,19,8)^G,
(1,22)(2,8)(5,9)(6,19)(10,18)(11,21)(13,14)(17,20)^G,
(1,9,2)(3,15,16)(4,7,12)(5,8,22)(10,14,20)(13,17,18)^G,
(1,8,9,22,2,5)(3,16,15)(4,12,7)(6,19)(10,17,14,18,20,13)(11,21)^G,
(1,4,21,10,16)(2,3,15,11,17)(5,13,14,18,8)(6,20,12,22,9)^G,
(1,20)(2,10,3,7)(5,18,15,22)(6,14,16,19)(8,9,21,17)(12,13)^G,
(1,12,20,13)(2,6,10,14,3,16,7,19)(4,11)(5,21,18,17,15,8,22,9)^G,
(1,15,6,13,5,9,3,20,4,11,19)(2,8,16,21,7,10,22,18,17,12,14)^G,
(1,19,11,4,20,3,9,5,13,6,15)(2,14,12,17,18,22,10,7,21,16,8)^G,
(1,14,22,13)(2,19,8,6)(3,7)(4,15)(5,20,9,17)(10,21,18,11)^G ]
```

So the conjugacy classes for M_{22} are:

C_o=[1^{22}] =[e] , [a] is the conjugacy class contains a.

C_1=[1^6 , 2^8] = [(1,22)(2,8)(5,9)(6,19)(10,18)(11,21)(13,14)(17,20)]

C_2=[1^4 , 3^6] = [(1,9,2)(3,15,16)(4,7,12)(5,8,22)(10,14,20)(13,17,18)]

C_3=[1^2 , 2^2 , 4^4] = [(1,14,22,13)(2,19,8,6)(3,7)(4,15)(5,20,9,17)(10,21,18,11)]

C_4=[1^2 , 2^2 , 4^4] = [(1,20)(2,10,3,7)(5,18,15,22)(6,14,16,19)(8,9,21,17)(12,13)]

C_5=[1^2 , 5^4] = [(1,4,21,10,16)(2,3,15,11,17)(5,13,14,18,8)(6,20,12,22,9)]

C_6=[2^2,3^2, 6^2] = [(1,8,9,22,2,5)(3,16,15)(4,12,7)(6,19)(10,17,14,18,20,13)(11,21)]

C_7=[1 , 7^3] = [(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)]

C_8=[1 , 7^3] = [(1,11,3,10,6,4,15)(2,21,17,22,16,12,9)(5,13,18,7,14,19,8)]

C_9=[2 , 4 , 8^2] = [(1,12,20,13)(2,6,10,14,3,16,7,19)(4,11)(5,21,18,17,15,8,22,9)]

C_{10}=[11^2] = [(1,15,6,13,5,9,3,20,4,11,19)(2,8,16,21,7,10,22,18,17,12,14)]

C_{11}=[11^2] = [(1,19,11,4,20,3,9,5,13,6,15)(2,14,12,17,18,22,10,7,21,16,8)]

3.2 Power of Conjugacy Classes in M_{22}:

This section devoted to write the following algorithm, algorithm power, by using GAP program to compute the powers of the conjugacy classes of M_{22} .

Algorithm 2 (The Algorithm Power):

```
gap> dd:= [ conjugate classes];;
gap> prod:= ();
()
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(The first power)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(The second power)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(and so on)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(Identity)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
()
```

For example ,

Let $\alpha \in C_7=[1,7^3]$, where $[\alpha]$= [(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)], we find the power of $[\alpha]$ by algorithm above in program GAP .

```
gap> dd:= [(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13) ];;
gap> prod:= ();
()
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,4,10,11,15,6,3)(2,12,22,21,9,16,17)(5,19,7,13,8,14,18)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,6,11,4,3,15,10)(2,16,21,12,17,9,22)(5,14,13,19,18,8,7)
gap> for p in pp do prod:= prod * p;
> od;
gap> prod; (1,10,15,3,4,11,6)(2,22,9,17,12,21,16)(5,7,8,18,19,13,14)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,3,6,15,11,10,4)(2,17,16,9,21,22,12)(5,18,14,8,13,7,19)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,11,3,10,6,4,15)(2,21,17,22,16,12,9)(5,13,18,7,14,19,8)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
()
```

To compute powers of [**α**] and which conjugacy classes to belong, we use the algorithm below in program GAP .

Algorithm 3:

```
gap> s22:= Group( (1,2), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22) );
gap> M22:= Group( (1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22),
(1,4,5,9,3)(2,8,10,7,6)(12,15,16,20,14)(13,19,21,18,17),(11,22)(1,21)(2,10,8,6)(12,14,16,20)
(4,17,3,13)(5,19,9,18) );
gap> ccl:= ConjugacyClasses( M22 );
gap> class := ConjugacyClass(M22,( any class);
gap> (The first power) in class;
true  or  false
gap> (The second power) in class;
true  or  false
```

```
gap> (and so on) in class;
true  or  false
```

$\alpha \in C_7 = [1,7^3]$ = [(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)]

$[\alpha^1] = C_7$

$[\alpha^2]$=[(1,4,10,11,15,6,3)(2,12,22,21,9,16,17)(5,19,7,13,8,14,18)] $\in C_7$

$[\alpha^3]$=[(1,6,11,4,3,15,10)(2,16,21,12,17,9,22)(5,14,13,19,18,8,7)] $\in C_8$

$[\alpha^4]$=[(1,10,15,3,4,11,6)(2,22,9,17,12,21,16)(5,7,8,18,19,13,14)] $\in C_7$

$[\alpha^5]$=[(1,3,6,15,11,10,4)(2,17,16,9,21,22,12)(5,18,14,8,13,7,19)] $\in C_8$

$[\alpha^6]$=[(1,11,3,10,6,4,15)(2,21,17,22,16,12,9)(5,13,18,7,14,19,8)] $\in C_8$

$[\alpha^7] = C_0$

So for **[α]** above we have:

gap>class:=ConjugacyClass(M22,(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13));
(1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13)^G

gap> (1,15,4,6,10,3,11)(2,9,12,16,22,17,21)(5,8,19,14,7,18,13) in class;
true

gap> (1,4,10,11,15,6,3)(2,12,22,21,9,16,17)(5,19,7,13,8,14,18) in class;
true

gap> (1,6,11,4,3,15,10)(2,16,21,12,17,9,22)(5,14,13,19,18,8,7) in class;
false

gap> (1,10,15,3,4,11,6)(2,22,9,17,12,21,16)(5,7,8,18,19,13,14) in class;
true

gap> (1,3,6,15,11,10,4)(2,17,16,9,21,22,12)(5,18,14,8,13,7,19) in class;
false

gap> (1,11,3,10,6,4,15)(2,21,17,22,16,12,9)(5,13,18,7,14,19,8) in class;
false

Compute the other conjugacy classes for M_{22} we have the powers of these conjugacy classes as in Table (1).

Table1: The powers of M_{22}

The conjugate class		1	2	3	4	5	6	7	8	9	10	11
C_0	$[1^{22}]$	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0
C_1	$[1^6,2^8]$	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1
C_2	$[1^4,3^6]$	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2
C_3	$[1^2,2^2,4^4]$	C_3	C_1	C_3	C_0	C_3	C_1	C_3	C_0	C_3	C_1	C_3
C_4	$[1^2,2^2,4^4]$	C_4	C_1	C_4	C_0	C_4	C_1	C_4	C_0	C_4	C_1	C_4
C_5	$[1^2,5^4]$	C_5	C_5	C_5	C_5	C_0	C_5	C_5	C_5	C_5	C_0	C_5
C_6	$[2^2,3^2,6^2]$	C_6	C_2	C_1	C_2	C_6	C_0	C_6	C_2	C_1	C_2	C_6
C_7	$[1,7^3]$	C_7	C_7	C_8	C_7	C_8	C_8	C_0	C_7	C_7	C_8	C_7
C_8	$[1,7^3]$	C_8	C_8	C_7	C_8	C_7	C_7	C_0	C_8	C_8	C_7	C_8
C_9	$[2,4,8^2]$	C_9	C_4	C_9	C_1	C_9	C_4	C_9	C_0	C_9	C_4	C_9
C_{10}	$[11^2]$	C_{10}	C_{11}	C_{10}	C_{10}	C_{10}	C_{11}	C_{11}	C_{11}	C_{10}	C_{11}	C_0
C_{11}	$[11^2]$	C_{11}	C_{10}	C_{11}	C_{11}	C_{11}	C_{10}	C_{10}	C_{10}	C_{11}	C_{10}	C_0

3.3 The Roots of M_{22}:

From the Table (1), the powers x^n of the conjugacy classes for M_{22}, we can compute the roots of M_{22} as in Table (2), that represents the solution of class equation $x^n = a$, where x is $n\,th$ roots of a .

Table2: The roots of M_{22}

Powers / ConjugacyClasses		1	2	3	4	5	6	7	8	9	10	11
C_0	$[1^{22}]$	C_0	C_0,C_1	C_0,C_2	C_0,C_1 C_3,C_4	C_0,C_5	C_0,C_1 C_2,C_6	C_0,C_7 C_8	C_0,C_1,C_3 C_4,C_9	C_0,C_2	C_0,C_1 C_5	C_0,C_{10} C_{11}
C_1	$[1^6,2^8]$	C_1	C_3,C_4	C_1,C_6	C_9	C_1	C_3,C_4	C_1		C_1,C_6	C_3,C_4	C_1
C_2	$[1^4,3^6]$	C_2	C_2,C_6		C_2,C_6	C_2		C_2	C_2,C_6		C_2,C_6	C_2
C_3	$[1^2,2^2,4^4]$	C_3		C_3		C_3		C_3		C_3		C_3
C_4	$[1^2,2^2,4^4]$	C_4	C_9	C_4		C_4	C_9	C_4		C_4	C_9	C_4
C_5	$[1^2,5^4]$	C_5	C_5	C_5	C_5		C_5	C_5	C_5	C_5		C_5
C_6	$[2^2,3^2,6^2]$	C_6				C_6		C_6				C_6
C_7	$[1,7^3]$	C_7	C_7	C_8	C_7	C_8	C_8		C_7	C_7	C_8	C_7
C_8	$[1,7^3]$	C_8	C_8	C_7	C_8	C_7	C_7		C_8	C_8	C_7	C_8
C_9	$[2,4,8^2]$	C_9		C_9		C_9		C_9		C_9		C_9
C_{10}	$[11^2]$	C_{10}	C_{11}	C_{10}	C_{10}	C_{10}	C_{11}	C_{11}	C_{11}	C_{10}	C_{11}	

C_{11}	$[11^2]$	C_{11}	C_{10}	C_{11}	C_{11}	C_{11}	C_{10}	C_{10}	C_{10}	C_{11}	C_{10}	

4. The Mathieu Group M_{23} [3]:

M_{23} is one of the Mathieu groups, a sporadic simple group of order $10200960=2^7.3^2.5.7.11.23$ M_{23} can be given as the subgroup of S_{23} generated by the permutations.

D=(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23)

E=(3,17,10,7,9)(5,4,13,14,19)(11,12,23,8,18)(21,16,15,20,22)

M_{23}=< D, E >.

4.1 Conjugacy Classes of M_{23}:

We can find conjugacy classes of M_{23} a subgroup from the symmetric group S_{23} by using this algorithm below in program GAP.

```
gap> s23:= Group( (1,2), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23) );
Group([ (1,2), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23) ])
gap> M23:= Group( (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
(3,17,10,7,9)(5,4,13,14,19)(11,12,23,8,18)(21,16,15,20,22) )
gap> CCl:= ConjugacyClasses( M23 );
[ ()^G,
(2,5)(3,23)(4,20)(6,18)(7,8)(9,10)(13,16)(15,22)^G,
(1,17,19,11,21,14,12)(2,15,13,3,9,8,6)(5,22,16,23,10,7,18)^G,
(1,12,14,21,11,19,17)(2,6,8,9,3,13,15)(5,18,7,10,23,16,22)^G,
(1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)^G,
(1,14,11,17,12,21,19)(2,7,3,22,6,10,13,5,8,23,15,18,9,16)(4,20)^G,
(1,20,6,23,8,22,11,2,3,10,4,21,14,19,18,7,15,5,9,17,16,13,12)^G,
(1,12,13,16,17,9,5,15,7,18,19,14,21,4,10,3,2,11,22,8,23,6,20)^G,
(1,14,19,20,18)(3,6,22,13,5)(4,12,7,23,11)(8,9,15,10,17)^G,
(2,22,6)(3,7,15)(4,20,9)(5,16,23)(8,13,18)(11,17,21)^G,
(1,14)(2,6,22)(3,17,7,21,15,11)(4,9,20)(5,8,16,13,23,18)(10,12)^G,
(1,4,22,20,23,3,14,12,13,18,11,6,19,7,5)(2,16,21)(8,9,15,10,17)^G,
(1,6,12,20,5,11,14,22,7,18,3,4,19,13,23)(2,21,16)(8,9,15,10,17)^G,
(1,7,20,17,6,9,22,18,11,3,4)(5,12,15,14,8,23,13,10,19,21,16)^G,
(1,4,3,11,18,22,9,6,17,20,7)(5,16,21,19,10,13,23,8,14,15,12)^G,
(1,6)(2,22,14,12)(3,8,13,23)(5,9,15,11)(7,16)(17,18,21,19)^G,
(1,16,6,7)(2,21,22,19,14,17,12,18)(3,9,8,15,13,11,23,5)(4,20)^G ]
```

So the conjugacy classes for M_{23} are:

C_o=[1^{23}] =[e] , [a] is the conjugacy class contains a.

C_1=[1^7 , 2^8] = [(2,5)(3,23)(4,20)(6,18)(7,8)(9,10)(13,16)(15,22)]

C_2=[1^5 , 3^6] = [(2,22,6)(3,7,15)(4,20,9)(5,16,23)(8,13,18)(11,17,21)]

C_3=[1^3 , 2^2 , 4^4] = [(1,6)(2,22,14,12)(3,8,13,23)(5,9,15,11)(7,16)(17,18,21,19)]

C_4=[1^3 , 5^4] = [(1,14,19,20,18)(3,6,22,13,5)(4,12,7,23,11)(8,9,15,10,17)]

C_5=[$1,2^2,3^2,6^2$] =[(1,14)(2,6,22)(3,17,7,21,15,11)(4,9,20)(5,8,16,13,23,18)(10,12)]

C_6=[1^2 , 7^3] = [(1,17,19,11,21,14,12)(2,15,13,3,9,8,6)(5,22,16,23,10,7,18)]

C_7=[1^2 , 7^3] = [(1,12,14,21,11,19,17)(2,6,8,9,3,13,15)(5,18,7,10,23,16,22)]

C_8=[$1,2,4,8^2$] = [(1,16,6,7)(2,21,22,19,14,17,12,18)(3,9,8,15,13,11,23,5)(4,20)]

C_9=[1 , 11^2] = [(1,7,20,17,6,9,22,18,11,3,4)(5,12,15,14,8,23,13,10,19,21,16)]

C_{10}=[1 , 11^2] = [(1,4,3,11,18,22,9,6,17,20,7)(5,16,21,19,10,13,23,8,14,15,12)]

C_{11}=[2,7,14] = [(1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)]

C_{12}=[2,7,14] = [(1,14,11,17,12,21,19)(2,7,3,22,6,10,13,5,8,23,15,18,9,16)(4,20)]

C_{13}=[3 ,5,15] = [(1,4,22,20,23,3,14,12,13,18,11,6,19,7,5)(2,16,21)(8,9,15,10,17)]

C_{14}=[3 ,5,15] = [(1,6,12,20,5,11,14,22,7,18,3,4,19,13,23)(2,21,16)(8,9,15,10,17)]

C_{15}=[23] = [(1,20,6,23,8,22,11,2,3,10,4,21,14,19,18,7,15,5,9,17,16,13,12)]

C_{16}=[23] = [(1,12,13,16,17,9,5,15,7,18,19,14,21,4,10,3,2,11,22,8,23,6,20)]

4.2 Powers of The Conjugate Class of M_{23}:

We can compute the powers of the conjugacy classes of M_{23} by using this algorithm below in program GAP.

```
gap> dd:= [ conjugacy classes];;
gap> prod:= ();
()
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(The first power)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;

(The second power)
gap> for d in dd do prod:= prod * d;
```

```
> od;
gap> prod;
(and so on)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
(Identity)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
()
```

For example that,

Let $\alpha \in C_{11}$=[2,7,14]

[α]= [(1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)], we find the power of [α] by Algorithm Power in program GAP.

```
gap> dd:= [ (1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20) ];;
gap> prod:= ();
()
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,17,19,11,21,14,12)(2,15,13,3,9,8,6)(5,22,16,23,10,7,18)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,14,11,17,12,21,19)(2,7,3,22,6,10,13,5,8,23,15,18,9,16)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,19,21,12,17,11,14)(2,13,9,6,15,3,8)(5,16,10,18,22,23,7)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,12,14,21,11,19,17)(2,18,8,10,3,16,15,5,6,7,9,23,13,22)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,11,12,19,14,17,21)(2,3,6,13,8,15,9)(5,23,18,16,7,22,10)
gap> for d in dd do prod:= prod *d;
> od;
gap> prod; (2,5)(3,23)(4,20)(6,18)(7,8)(9,10)(13,16)(15,22)
```

```
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,21,17,14,19,12,11)(2,9,15,8,13,6,3)(5,10,22,7,16,18,23)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,17,19,11,21,14,12)(2,22,13,23,9,7,6,5,15,16,3,10,8,18)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,14,11,17,12,21,19)(2,8,3,15,6,9,13)(5,7,23,22,18,10,16)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,19,21,12,17,11,14)(2,16,9,18,15,23,8,5,13,10,6,22,3,7)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,12,14,21,11,19,17)(2,6,8,9,3,13,15)(5,18,7,10,23,16,22)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod; (1,11,12,19,14,17,21)(2,23,6,16,8,22,9,5,3,18,13,7,15,10)(4,20)
gap> for d in dd do prod:= prod * d;
> od;
gap> prod;
()
```

To compute powers of [α] and which conjugacy classes to belong, we use the algorithm below in program GAP .

```
gap> s23:= Group( (1,2), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23) );
Group([ (1,2), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23) ])
gap> M23:= Group( (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
(3,17,10,7,9)(5,4,13,14,19)(11,12,23,8,18)(21,16,15,20,22) )
gap> ccl:= ConjugacyClasses( M23 );
gap> class := ConjugacyClass(M23,( any class);
gap> (The first power) in class;
true  or  false
gap> (The second power) in class;
true  or  false
gap> (and so on) in class;
true  or  false
```

$\alpha \in C_{11}$=[2,7,14] = [(1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)]

[α^1]= C_{11}

[α^2] = [(1,17,19,11,21,14,12)(2,15,13,3,9,8,6)(5,22,16,23,10,7,18)] $\in C_6$

[α^3]= [(1,14,11,17,12,21,19)(2,7,3,22,6,10,13,5,8,23,15,18,9,16)(4,20)] $\in C_{12}$

[α^4] = [(1,19,21,12,17,11,14)(2,13,9,6,15,3,8)(5,16,10,18,22,23,7)] $\in C_6$

[α^5] = [(1,12,14,21,11,19,17)(2,18,8,10,3,16,15,5,6,7,9,23,13,22)(4,20)] $\in C_{12}$

[α^6] = [(1,11,12,19,14,17,21)(2,3,6,13,8,15,9)(5,23,18,16,7,22,10)] $\in C_6$

[α^7] = [(2,5)(3,23)(4,20)(6,18)(7,8)(9,10)(13,16)(15,22)] $\in C_1$

[α^8] = [(1,21,17,14,19,12,11)(2,9,15,8,13,6,3)(5,10,22,7,16,18,23)] $\in C_6$

[α^9] = [(1,17,19,11,21,14,12)(2,22,13,23,9,7,6,5,15,16,3,10,8,18)(4,20)] $\in C_{11}$

[α^{10}] = [(1,14,11,17,12,21,19)(2,8,3,15,6,9,13)(5,7,23,22,18,10,16)] $\in C_6$

[α^{11}] = [(1,19,21,12,17,11,14)(2,16,9,18,15,23,8,5,13,10,6,22,3,7)(4,20)] $\in C_{11}$

[α^{12}] = [(1,12,14,21,11,19,17)(2,6,8,9,3,13,15)(5,18,7,10,23,16,22)] $\in C_7$

[α^{13}] = [(1,11,12,19,14,17,21)(2,23,6,16,8,22,9,5,3,18,13,7,15,10)(4,20)] $\in C_{12}$

[α^{14}] = C_0

So for **[α]** above we have:

```
gap>class:=ConjugacyClass(M23,(1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)
(4,20)); (1,21,17,14,19,12,11)(2,10,15,7,13,18,3,5,9,22,8,16,6,23)(4,20)^G
gap> (1,17,19,11,21,14,12)(2,15,13,3,9,8,6)(5,22,16,23,10,7,18) in class;
false
gap> (1,14,11,17,12,21,19)(2,7,3,22,6,10,13,5,8,23,15,18,9,16)(4,20) in class;
false
gap> (1,19,21,12,17,11,14)(2,13,9,6,15,3,8)(5,16,10,18,22,23,7) in class;
false
gap> (1,12,14,21,11,19,17)(2,18,8,10,3,16,15,5,6,7,9,23,13,22)(4,20) in class;
false
gap> (1,11,12,19,14,17,21)(2,3,6,13,8,15,9)(5,23,18,16,7,22,10) in class;
false
gap> (2,5)(3,23)(4,20)(6,18)(7,8)(9,10)(13,16)(15,22) in class;
false
gap> (1,21,17,14,19,12,11)(2,9,15,8,13,6,3)(5,10,22,7,16,18,23) in class;
false
gap> (1,17,19,11,21,14,12)(2,22,13,23,9,7,6,5,15,16,3,10,8,18)(4,20) in class;
true
```

gap> (1,14,11,17,12,21,19)(2,8,3,15,6,9,13)(5,7,23,22,18,10,16) in class;
false
gap> (1,19,21,12,17,11,14)(2,16,9,18,15,23,8,5,13,10,6,22,3,7)(4,20) in class;
true
gap> (1,12,14,21,11,19,17)(2,6,8,9,3,13,15)(5,18,7,10,23,16,22) in class;
false
gap> (1,11,12,19,14,17,21)(2,23,6,16,8,22,9,5,3,18,13,7,15,10)(4,20) in class;
false

Compute the other conjugacy classes for M_{23} we have the powers of these conjugacy classes as in Table (3).

Table 3: The powers of M_{23}

The conjugate class		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
C_0	$[1^{23}]$	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0
C_1	$[1^7,2^8]$	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1	C_0	C_1
C_2	$[1^5,3^6]$	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2	C_0	C_2	C_2
C_3	$[1^3,2^2,4^4]$	C_3	C_1	C_3	C_0	C_3	C_1	C_3	C_0	C_3	C_1	C_3	C_0	C_3	C_1	C_3	C_0	C_3	C_1	C_3	C_0	C_3	C_1	C_3
C_4	$[1^3,5^4]$	C_4	C_4	C_4	C_4	C_0	C_4	C_4	C_4	C_4	C_0	C_4	C_4	C_4	C_4	C_0	C_4	C_4	C_4	C_4	C_0	C_4	C_4	C_4
C_5	$[1,2^2,3^2,6^2]$	C_5	C_2	C_1	C_2	C_5	C_0	C_5	C_2	C_1	C_2	C_5	C_0	C_5	C_2	C_1	C_2	C_5	C_0	C_5	C_2	C_1	C_2	C_5
C_6	$[1^2,7^3]$	C_6	C_6	C_7	C_6	C_7	C_7	C_0	C_6	C_6	C_7	C_6	C_7	C_7	C_0	C_6	C_6	C_7	C_6	C_7	C_7	C_0	C_6	C_6
C_7	$[1^2,7^3]$	C_7	C_7	C_6	C_7	C_6	C_6	C_0	C_7	C_7	C_6	C_7	C_6	C_6	C_0	C_7	C_7	C_6	C_7	C_6	C_6	C_0	C_7	C_7
C_8	$[1,2,4,8^2]$	C_8	C_3	C_8	C_1	C_8	C_3	C_8	C_0	C_8	C_3	C_8	C_1	C_8	C_3	C_8	C_0	C_8	C_3	C_8	C_1	C_8	C_3	C_8
C_9	$[1,11^2]$	C_9	C_{10}	C_9	C_9	C_9	C_{10}	C_{10}	C_{10}	C_9	C_{10}	C_0	C_9	C_{10}	C_9	C_9	C_9	C_{10}	C_{10}	C_{10}	C_9	C_{10}	C_0	C_9

C_{10}	$[1,11^2]$	C_{10}	C_9	C_{10}	C_{10}	C_{10}	C_9	C_9	C_9	C_{10}	C_9	C_0	C_{10}	C_9	C_{10}	C_{10}	C_{10}	C_9	C_9	C_9	C_{10}	C_9	C_0	C_{10}
C_{11}	[2,7,14]	C_{11}	C_6	C_{12}	C_6	C_{12}	C_6	C_1	C_6	C_{11}	C_6	C_{11}	C_7	C_{12}	C_0	C_{11}	C_6	C_{12}	C_6	C_{12}	C_6	C_1	C_6	C_{11}
C_{12}	[2,7,14]	C_{12}	C_7	C_{11}	C_7	C_{11}	C_7	C_1	C_7	C_{12}	C_6	C_{12}	C_7	C_{11}	C_0	C_{12}	C_7	C_{11}	C_7	C_{11}	C_7	C_1	C_7	C_{12}
C_{13}	[3,5,15]	C_{13}	C_{13}	C_4	C_{13}	C_2	C_4	C_{14}	C_{13}	C_4	C_2	C_{14}	C_4	C_{14}	C_{14}	C_0	C_{13}	C_{13}	C_4	C_{13}	C_2	C_4	C_{14}	C_{13}
C_{14}	[3,5,15]	C_{14}	C_{14}	C_4	C_{14}	C_2	C_4	C_{13}	C_{14}	C_4	C_2	C_{13}	C_4	C_{13}	C_{13}	C_0	C_{14}	C_{14}	C_4	C_{14}	C_2	C_4	C_{13}	C_{14}
C_{15}	[23]	C_{15}	C_{15}	C_{15}	C_{15}	C_{16}	C_{15}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{16}	C_{15}	C_{16}	C_{16}	C_{16}	C_{16}	C_0
C_{16}	[23]	C_{16}	C_{16}	C_{16}	C_{16}	C_{15}	C_{16}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{15}	C_{16}	C_{15}	C_{15}	C_{15}	C_{15}	C_0

4.3 The Roots of M_{23}

From the Table (3), the powers x^n of the conjugacy classes for M_{23},we can compute the root of M_{23} as in Table (4) that represents the solution of class equation $x^n = a$, where x is $n\,th$ roots of a .

Table4: The roots of M_{23}

powers / ConjugateClasses		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
C_0	$[1^{23}]$	C_0	C_0 C_1	C_0 C_2	C_0 C_1 C_3	C_0 C_4	C_0 C_1 C_2 C_5	C_0 C_6 C_7	C_0 C_1 C_3 C_8	C_0 C_2	C_0 C_1 C_4	C_0 C_9 C_{10}	C_0 C_1 C_2 C_3 C_5	C_0	C_0 C_1 C_6 C_7 C_{11} C_{12}	C_0 C_2 C_4 C_{13} C_{14}	C_0 C_1 C_3 C_8	C_0	C_0 C_1 C_2 C_5	C_0	C_0 C_1 C_3 C_4	C_0 C_2 C_6 C_7	C_0 C_1 C_9 C_{10}	C_0 C_{15} C_{16}

C_1	$[1^7,2^8]$	C_1	C_3	C_1 C_5	C_8	C_1	C_3	C_1 C_{11} C_{12}		C_1 C_5	C_3	C_1	C_8	C_1	C_3	C_1 C_5		C_1	C_3	C_1	C_8	C_1 C_5 C_{11} C_{12}	C_3	C_1
C_2	$[1^5,3^6]$	C_2	C_2 C_5		C_2 C_5	C_2 C_{13} C_{14}		C_2	C_2 C_5		C_2 C_5 C_{13} C_{14}	C_2		C_2	C_2 C_5		C_2 C_5	C_2		C_2	C_2 C_5 C_{13} C_{14}		C_2 C_5	C_2
C_3	$[1^3,2^2,4^4]$	C_3	C_8	C_3		C_3	C_8	C_3		C_3	C_8	C_3		C_3	C_8	C_3		C_3	C_8	C_3		C_3	C_8	C_3
C_4	$[1^3,5^4]$	C_4	C_4	C_4 C_{13} C_{14}	C_4		C_4 C_{13} C_{14}	C_4	C_4	C_4 C_{13} C_{14}		C_4	C_4 C_{13} C_{14}	C_4	C_4		C_4	C_4	C_4 C_{13} C_{14}	C_4		C_4 C_{13} C_{14}	C_4	C_4
C_5	$[1,2^2,3^2,6^2]$	C_5				C_5		C_5				C_5		C_5				C_5		C_5				C_5
C_6	$[1^2,7^3]$	C_6	C_6 C_{11}	C_7	C_6 C_{11}	C_7	C_7 C_{11}		C_6 C_{11}	C_6	C_7 C_{11} C_{12}	C_6	C_7	C_7		C_6	C_6 C_{11}	C_7	C_6 C_{11}	C_7	C_7 C_{11}		C_6 C_{11}	C_6
C_7	$[1^2,7^3]$	C_7	C_7 C_{12}	C_6	C_7 C_{12}	C_6	C_6 C_{12}		C_7 C_{12}	C_7	C_6	C_7	C_6 C_{11} C_{12}	C_6		C_7	C_7 C_{12}	C_6	C_7 C_{12}	C_6	C_6 C_{12}		C_7 C_{12}	C_7
C_8	$[1,2,4,8^2]$	C_8		C_8		C_8		C_8		C_8		C_8		C_8		C_8		C_8		C_8		C_8		C_8
C_9	$[1,11^2]$	C_9	C_{10}	C_9	C_9	C_9	C_{10}	C_{10}	C_{10}	C_9	C_{10}		C_9	C_{10}	C_9	C_9	C_9	C_{10}	C_{10}	C_{10}	C_9	C_{10}		C_9
C_{10}	$[1,11^2]$	C_{10}	C_9	C_{10}	C_{10}	C_{10}	C_9	C_9	C_9	C_{10}	C_9		C_{10}	C_9	C_{10}	C_{10}	C_{10}	C_9	C_9	C_9	C_{10}	C_9		C_{10}

C_{11}	[2,7,14]	C_{11}		C_{12}		C_{12}				C_{11}		C_{11}		C_{12}		C_{11}		C_{12}		C_{12}				C_{11}
C_{12}	[2,7,14]	C_{12}		C_{11}		C_{11}				C_{12}		C_{12}		C_{11}		C_{12}		C_{11}		C_{11}				C_{12}
C_{13}	[3,5,15]	C_{13}	C_{13}		C_{13}			C_{14}	C_{13}			C_{14}		C_{14}	C_{14}		C_{13}	C_{13}		C_{13}			C_{14}	C_{13}
C_{14}	[3,5,15]	C_{14}	C_{14}		C_{14}			C_{13}	C_{14}			C_{13}		C_{13}	C_{13}		C_{14}	C_{14}		C_{14}			C_{13}	C_{14}
C_{15}	[23]	C_{15}	C_{15}	C_{15}	C_{15}	C_{16}	C_{15}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{16}	C_{15}	C_{16}	C_{16}	C_{16}	C_{16}	
C_{16}	[23]	C_{16}	C_{16}	C_{16}	C_{16}	C_{15}	C_{16}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{16}	C_{15}	C_{15}	C_{16}	C_{15}	C_{16}	C_{15}	C_{15}	C_{15}	C_{15}	

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حل معادلة فروبينوس في زمر الماثيو الكبيرة

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المستخلص:

في هذا البحث ، تم دراسة حلول معادلة $x^n = a$ فروبينوس في مجموعتي الماثيوالكبيرة M_{22} و M_{23} حيث n هي عدد صحيح موجب ، و x هي صفوف ترافق. علاوة على ذلك ، تم توفير ثلاث خوارزميات في نظام GAP لحساب عدد الحلول والحلول نفسها (إن وجدت) لهذه المعادلة.