Solution of Periodic Hyperbolic Differential Problem with Initial Control Function

Z.A. Al Ameer, S.Q. Hasan*

Department of Mathematics, College of Science, University of AL-Mustansiriya, Baghdad, Iraq

*Corresponding author E-mail: zynbaldlal@gmail.com

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ABSTRACT

This work investigates the problem of a periodic hyperbolic equation formulated in a bounded domain with a convection term and weakly sourced nonlinear functions. The purpose of this work is to analyze the existence of solutions to the control initial problem using ideas of hyperbolic and periodicity of first derivatives and control starting conditions, as well as some properties of some inequalities offered to investigate this topic.

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1. Introduction

In this section, the periodic hyperbolic deferential initial control equation presented, as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + D_i\left(a_{ij}(t,x,\mathrm{u})D_j u\right) - g(x,t,u) = f(x,t) & & (1) \\ \dfrac{\partial u}{\partial n} = 0, & (n,t) \in \partial\Omega \times (0,T) & (2) \\ u(x,0) = u_1(x), & x \in \Omega & (3) \\ u'(x,0) = u'(x,T)\,, & x \in \Omega. & (4) \end{cases}$$

And Ω is in a bounded domain R^n with $\partial\Omega$ as smooth boundary, the outward normal derivative on $\partial\Omega$ denotes by $\frac{\partial}{\partial n}$, $Q_w = \Omega \times (0,T)$. The diffusion terms $D_i\left(a_{ij}(x,t,u)D_j u\right)$ represent the effect of dispersion description in many articles such as [1],[2], and $u_1(x)$ is nonnegative control function and satisfy $u_1 \in W_0^{1,2}(\Omega)$.

Assume the following conditions:

a. $a_{ij}(.,.,u) = a_{ij}(.,.,u) \epsilon C_w(\bar{Q}_w)$, and 0<$\lambda < \Lambda$ are two constants $\ni \lambda|\xi|^2 \leq a_{ij}(x,t,u)\xi_i\xi_j \leq \Lambda|\xi|^2$, the class of functions which are continuous in $\bar{\Omega} \times$ R is denoted by $C_w(\bar{Q}_w)$and of $w-$ periodic with respect to t. Furthermore, the continuity of $a_{ij}(.,.,u)$ respected to u.

b. $g(x,t,u)$ is Holder continuous in $\bar{\Omega} \times$ R $\times$ R, periodic in t with a period T and satisfied $g(x,t,u)u \leq b_0|u|^{\alpha+1}$ with constant $b_0 \geq 0$ and 0$\leq \alpha \leq 1$.

c. $f(x,t) \epsilon C_T(\bar{Q}_w) \cap L^\infty\left(0,T;W_0^{1,\infty}(\Omega)\right), f(x,t) > 0$ for $\Omega \times$ R, where $C_T(\bar{Q}_w)$ be the set of functions are continuous in $\bar{\Omega} \times$ R and $w-$ periodic with respect to t.

Many studies have been published on many forms of hyperbolic differential equations with various features and noteworthy results, such as [3], which investigates multipoint boundary value problems with nonlocal beginning conditions for hyperbolic deferential problems using various techniques. In [4,] the Impulsive System with Periodic Problem for a Hyperbolic Equation has been thoroughly studied. Relationship of periodic problem for the system of hyperbolic equations with finite time delay and the family of periodic problems for the system of ordinary differential equations with finite delay is established in [5], has been studied periodic solutions in terms of

Jacobi elliptic functions as well as the corresponding hyperbolic soliton solutions in [6], the optimal boundary control problems of the turnpike phenomenon corresponding to hyperbolic systems have been studied and computed. Investigated a degenerate hyperbolic wave equation with Neumann boundary conditions and bilinear control in their paper. In [7], they have a non-hyperbolic singularity at the origin of arbitrary degeneracy and they studied a Boundary control of hyperbolic system(1)–(4). In [9], the diffusion-reaction equation is transformed into a 2 ×2 system of coupled parabolic PDEs with exotic boundary conditions. Finally in [13], studied the behavior of the solution of the Cauchy problem for both homogeneous and inhomogeneous hyperbolic equations, as well as the behavior of the solution of a mixed initial-boundary value problem for the same equations, are studied. In this paper, the existence of the solution of periodic hyperbolic problem with control boundary functions (1) - (4) is shown, and it is shown that the solution is uniformly bounded. The existence under some conditions is sufficiently to grantee the nontrivial periodic solutions of the system (1)–(4). Hence, the following definition to the solutions of the problem (1) – (4) is giving in the section 2.

2. Main results

In this section, certain results are presented that explain the existence of a solution to the periodic hyperbolic deferential initial control equation, which requires the following generalized solution specification.

Definition (2.1):

The solution of Eqs. (1)- (4) is a nonnegative continuous vector-valued function (u, v) that is satisfied, for any $0 \le \tau < T$, $\varphi \in C^1(\overline{Q_T})$ with $\varphi \mid \partial\Omega \times [0, \tau) = 0$

$$\iint_{Q_T} u\frac{\partial^2\varphi}{\partial t^2} + D_i\left(a_{ij}(x, t, u)D_j u\right) + g(x, u)\varphi dx\, dt$$

$$= \int_{\Omega} u(x, t)\, \varphi(x, \tau)dx - \int_{\Omega} u_1(x)\varphi_1(x, 0)dx \qquad (5)$$

where $Q_T = \Omega \times (0, T)$.

Theorem (2.1):

Consider following the periodic hyperbolic deferential initial control equations (1-4) and $\sigma\epsilon[0,1]$, then$\|u(t)\|_{L^{\infty}(Q_T)} < \breve{R}$ where $\breve{R}$ is a positive constant independent of σ.

Proof:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2}\ \mathrm{u}^{\mathrm{m}+2} + D_i(a_{ij}(t,x,u)D_j u\, u^{m+2} = -u^{m+2}g(x,u) + u^{m+2}f(x,t)$$

From $\frac{1}{(m+2)(m+3)}\ \frac{\partial^2 u^{m+3}}{\partial t^2}$

$$= \frac{1}{m+2}\ [(m+2)\ u^{m+1}\left(\frac{\partial u}{\partial t}\right)^2 + u^{m+2}\ \frac{\partial^2 u}{\partial t^2}]$$

$$= u^{m+1}\ \nabla u + \frac{u^{m+2}}{m+2}\ \frac{\partial^2 u}{\partial t^2}$$

Thus

$$\frac{1}{(m+3)}\ \frac{\partial^2 u}{\partial t^2}\ u^{m+3} - (m+2)\ u^{m+1}\ \nabla u = u^{m+2}\ \frac{\partial^2 u}{\partial t^2}$$

Therefor

$$\frac{1}{m+3}\ \frac{\partial^2 u}{\partial t^2}\ u^{m+3} - (m+2)\ u^{m+1}\ \nabla u + (a_{ij}(t,x,u)D_j u\ \ D_i u^{m+2}$$

$$= -u^{m+2}g(x,u) + u^{m+2}f(x,t)$$

$$\frac{1}{m+3}\ \frac{\partial^2 u}{\partial t^2}\ u^{m+3} - (m+2)\ u^{m+1}\ \nabla u + (a_{ij}(t,x,u)D_j u\ (m+2)\ u_i^{m+1}\ \ D_i u$$

$$= -u^{m+2}g(x,u) + u^{m+2}f(x,t)$$

$$\frac{1}{m+2}\ \frac{\partial^2 u}{\partial t^2}\ u^{m+2} - (m+2)\ u^{m+1}\ \nabla u + a_{ij}(t,x,u)(m+2)|\nabla u|^2\ \ u_i^{m+1}$$

$$= -u^{m+2}g(x,u) + u^{m+2}f(x,t)$$

$$\frac{1}{m+2}\ \frac{\partial^2 u}{\partial t^2}\ u^{m+2} - (m+2)\ u^{m+1}\ \nabla u + a_{ij}(t,x,u)(m+2)\left|\nabla u\, u^{\frac{m+1}{2}}\right|^2$$

$$= -u^{m+2}g(x,u) + u^{m+2}f(x,t)$$

From

$$\frac{2}{m+3}\ \nabla\left(u^{\frac{m+1}{2}+1}\right) = u^{\frac{m+1}{2}}\ \ \nabla u$$

$\frac{1}{m+2}\frac{\partial^2 u}{\partial t^2}\; u^{m+2} - (m+2)\, u^{m+1}\,\nabla u + (\, a_{ij}(t,x,u)(m+2)\left|\frac{2}{m+3}\,\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$= -u^{m+2}g(x,u) + u^{m+2}f(x,t)$, thus

$\frac{1}{m+2}\frac{\partial^2 u}{\partial t^2}\; u^{m+2} - (m+2)\, u^{m+1}\,\nabla u + (\, a_{ij}(t,x,u)\frac{4(m+2)}{(m+3)^2}\left|\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$= -u^{m+2}g(x,u) + \; u^{m+2}f(x,t)$

$\frac{1}{m+2}\frac{\partial^2 u}{\partial t^2}\; u^{m+2} - (m+2)\, u^{m+1}\,\nabla u + (\, a_{ij}(t,x,u)\frac{4(m+2)}{(m+3)^2}\left|\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$\leq b_0\,|u|^{\propto+1}\,u^{m+2} + u^{m+2}f(x,t) \leq b_0\,|u|^{\propto+m+3} + |u|^{m+1}\,u(t)\,f(x,t)$

$\frac{1}{m+2}\frac{\partial^2 u}{\partial t^2}\; u^{m+2} - (m+2)\, u^{m+1}\,\nabla u + (\, a_{ij}(t,x,u)\frac{4(m+2)}{(m+3)^2}\left|\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$\leq b_0\,|u|^{\propto+m+3} + |u|^{m+1}\,u(t)\,f(x,t)$ (6)

$\frac{1}{m+2}\frac{\partial^2 u}{\partial t^2}\int_\Omega u^{m+2}dx - (m+2)\int_\Omega u^{m+1}\,\nabla u\, dx$

$-\int_\Omega(\, a_{ij}(t,x,u)\frac{4(m+2)}{(m+3)^2}\left|\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$\leq\; b_0\int_\Omega|u|^{\propto+m+3}\,dx + \int_\Omega|u|^{m+1}\,u(t)f(x,t)\,dx$ (7)

from

$\int_\Omega|u|^{m+1}\,u(t)f(x,t)\,dx \leq \int_\Omega u^{m+2}\,dx\,)^{\frac{m+1}{p+2}}\;\;(\int_\Omega f^{m+2}\,dx)^{\frac{1}{m+2}}$

$\frac{1}{m+2}\frac{d^2}{dt^2}\,\|u\|_{m+2}^{m+2} - (m+2)\int_\Omega u^{m+1}\,\nabla u\,dx - \frac{4(m+2)\,\lambda}{(m+3)^2}\int_\Omega\left|\nabla(u^{\frac{m+1}{2}+1})^2\right|^2 dx$

$\leq\; C_2\,(\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$, to get

$\frac{1}{m+2}\frac{d^2}{dt^2}\,\|u\|_{m+2}^{m+2} - (m+2)\int_\Omega u^{m+1}\,\nabla u\,dx$

$+\frac{4(m+2)}{(m+3)^2}\int_\Omega(\, a_{ij}(t,x,u)\;\left|\nabla(u^{\frac{m+1}{2}+1})\right|^2$

$\leq\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+2})$,

$\frac{1}{m+2}\ \frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - (m+2)\int_\Omega \frac{1}{m+2}\nabla(u^{m+2}\ )\ dx + \frac{4(m+2)\ \lambda}{(m+3)^2}\int_\Omega \left|\nabla(u^{\frac{m+1}{2}+1})\right|^2 dx$

$\leq\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$

$\frac{1}{m+2}\ \frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - \|\nabla(u^{m+2})\| + \frac{4(m+2)\ \lambda}{(m+3)^2}\ \left\|\nabla(u^{\frac{m+1}{2}+1})^2\right\|_2^2$

$\leq\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$ (8)

$\frac{1}{m+2}\ \frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - \|\nabla(u^{m+2})\| + \frac{4(m+2)\ \lambda}{(m+3)^2}\ \left\|\nabla(u^{\frac{m+1}{2}+1})^2\right\|_2^2$

$\leq\ (m+2)\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$

$\frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - \|\nabla(u^{m+2})\| + C_1\ \left\|\nabla(u^{\frac{m+1}{2}+1})^2\right\|_2^2$

$\leq\ (m+2)\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$

$\frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - \|\nabla(u^{m+2})\| + C_1\ \|\nabla(u^{m+3})\|_2^2$

$\leq\ (m+2)\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$, set k as follows

$k = min\left\{\left\|(u^{\frac{m+1}{2}+1})^2\right\|_2^2, \|\nabla(u^{m+2})\|\ \right\}$

$\frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} - \|\nabla(u^{m+2})\| + C_1\ \|\nabla(u^{m+3})\|_2^2\ \leq\ (m+2)\ C_2\ (\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$

$\frac{d^2}{dt^2}\ \|u\|_{m+2}^{m+2} + (C_1 - 1)\|\nabla(u^{m+2})\| \leq (m+2)\ C_2(\|u\|_{\propto+m+3}^{\propto+m+3} + \|u\|_{m+2}^{m+1})$

Let $u_k = u^{\frac{m+1}{2}+1}$

If $0 < \alpha < 1$, to get

$\int_\Omega |u(t)|^{\propto+m+3} dx \leq\ (\int_\Omega |u(t)|^{m+2} dx\ )^{\frac{\propto+m+3}{\propto+m+3}}\ |\Omega|^{\frac{1-\propto}{m+2}}$

$\leq \max\left\{1, |\Omega|^{\frac{1}{2}}\right\} \|u\|_{m+2}^{\propto+m+3} \leq \max\left\{1, |\Omega|^{\frac{1}{2}}\right\} \|u\|_{m+2}^{(m+2)\propto} \|u\|_{m+2}^{(m+3)(1-\propto)}$

$$\leq \|u\|_{m+2}^{m+2} + C\|u\|_{m+2}^{m+3} \quad (9)$$

$\frac{d^2}{dt^2} \|u\|_{m+2}^{m+2} + (C_1 - 1)\|\nabla(u^{m+2})\|$

$$\leq (m+2)\, C_2(\|u\|_{m+2}^{m+2} + C\|u\|_{m+2}^{m+3} + \|u\|_{m+2}^{m+1}) \quad (10)$$

$\frac{d^2}{dt^2} \|u_k(t)\|_2^2 + (C_1 - 1)\|\nabla u_k(t)\|_2^2$

$\leq (m_k + 2)\, C_2(\|u_k(t)\|_2^2 + C\|u_k(t)\|^{\frac{2(m_k+3)}{m_k+2}} + \|u_k(t)\|^{\frac{2(m_k+1)}{m_k+2}})$

By the Gagliardo -Nirenberg inequality, to have

$$\|u_k(t)\|_2^2 \leq C\|\nabla u_k(t)\|_2^{\theta} \, \| u_k(t)\|_1^{1-\theta} \text{, where } \theta = \frac{N}{N+2} \in (0,1). \quad (11)$$

Noticing $\|u_k(t)\|_1 = \| u_k(t)\|_2^2$, by (3.10), to obtain

$\frac{d^2}{dt^2} \|u_k(t)\|_2^2 \leq (1 - C_1)\|\nabla u_k(t)\|_2^2 + (m_k + 2)C_2(\|u_k(t)\|_2^2$

$+C\|u_k(t)\|^{\frac{2(m_k+3)}{m_k+2}} + \|u_k(t)\|^{\frac{2(m_k+1)}{m_k+2}})$

$\frac{d^2}{dt^2} \|u_k(t)\|_2^2 \leq (1 - C_1)\| u_k(t)\|_2^{\frac{2}{\theta}} \|u_k(t)\|_1^{\frac{2(\theta-1)}{\theta}}$

$+(m_k + 2)C_2(\|u_k(t)\|_2^2 + C\|u_k(t)\|^{\frac{2(m_k+3)}{m_k+2}} + \|u_k(t)\|^{\frac{2(m_k+1)}{m_k+2}})$

$\frac{d^2}{dt^2} \|u_k(t)\|_2^2 \leq (1 - C_1)\| u_k(t)\|_2^{\frac{2}{\theta}} \|u_{k-1}(t)\|_2^{\frac{4(\theta-1)}{\theta}} + (m_k + 2)C_2(\|u_k(t)\|_2^2$

$$+C\|u_k(t)\|^{\frac{2(m_k+3)}{m_k+2}} + \|u_k(t)\|^{\frac{2(m_k+1)}{m_k+2}}) \quad (12)$$

Set

$\lambda_k = \max\{1, sup_t\|u_k(t)\|_2^2\}$, then

$$\frac{d^2}{dt^2}\|u_k(t)\|_2^2 \le \|u_k(t)\|^{\frac{2(p_k+3)}{p_k+2}}\Big\{(1-C_1)\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(p_k+3)}{p_k+2}}\lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} + (m_k +$$

$$2)C_2(\|u_k(t)\|_2^{\frac{2}{p_k+2}} + (m_k+2)C_2(C+1)\Big\}$$

(13)

By Young's inequality, for

$(m_k+2)\|u_k(t)\|^{\frac{2}{p_k+2}}$, to get that

$$ab \le \epsilon a^{p'} + \epsilon^{-\frac{q'}{p'}}\frac{1}{p'}\left(\frac{1}{p'}\right)^{\frac{q'}{p'}} b^{q'},$$

Where $p' > 0,\ a' > 1,\ \frac{1}{p'} + \frac{1}{q'} = 1$, with

$$a = \|u_k(t)\|_2^{\frac{2}{p_k+2}}, b = p_k + 2, \epsilon = \frac{1}{2}\lambda_{k-1}^{\frac{4(\theta-1)}{\theta}}$$

$$p' = \ell_k = \frac{p_k+2}{\theta} - p_k - 1$$

To get

$$(p_k+2)\|u_k(t)\|_2^{\frac{2}{p_k+2}} \le \frac{1}{2}\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(p_k+1)}{p_k+2}}\lambda_{k-1}^{\frac{4(\theta-1)}{\theta}} + C(p_k+2)^{\frac{\ell_k}{\ell_k-1}}\lambda_{k-1}^{\frac{4(1-\theta)}{\theta(\ell_k-1)}} \quad (14)$$

It is easy to see that $\lim_{k\to\infty}\ell_k = +\infty$

$$a_k = \frac{\ell_k}{\ell_k-1}, b_k = \frac{4(1-\theta)}{\theta(\ell_k-1)}$$

From (12), (13), to have that

$$\frac{d^2}{dt^2}\|u_k(t)\|_2^2 \le \|u_k(t)\|^{\frac{2(p_k+1)}{p_k+2}}\{(1-C_1)\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}}\lambda_{k-1}^{\frac{4(\theta-1)}{\theta}}$$

$$(m_k+2)^{a_k}C_2\lambda_{k-1}^{b_k} + (m_k+2)C_2(C+1)\}$$

Thus,

$$(m_k+2)\frac{d^2}{dt^2}\|u_k(t)\|_2^{\frac{2}{m_k+2}} \le (1-C_1)\,\|u_k(t)\|_2^{\frac{2}{\theta}-\frac{2(m_k+1)}{m_k+2}}\lambda^{\frac{4(\theta-1)}{\theta}}$$

$$+(m_k+2)^{a_k}C_2\,\lambda_{k-1}^{b_k}+(m_k+2)C_2(C+1) \quad (15)$$

The periodicity of $u'_k(t)$ implies that there exist t' such that$\|u'_k(t)\|_2$, then obtain

$$\|u_k(t)\|_2 \le \{(m_k+2)C_2(C+1)+(m_k+2)^{a_k}C_2\,\lambda_{k-1}^{b_k}\}^{\frac{1}{C_k}}$$

Where $C_k=\frac{2}{\theta}-\frac{2(p_k+1)}{p_k+2}=\frac{2\ell_k}{p_k+2}$

Since $\lambda_{k-1}\ge 1,(k=1,2,)$, it follows that

$$\|u_k(t)\|_2 \le \left\{C[(m_k+2)^{a_k}\,\lambda_{k-1}^{b_k+\frac{4(1-\theta)}{\theta}}\right\}^{\frac{1}{C_k}}$$

$$=\{C(m_k+2)^{a_k}\}^{\frac{m_k+2}{2\ell_k}}\,\lambda_{k-1}^{\frac{4(1-\theta)(m_k+2)}{2(\ell_{k-1})\theta}}$$

Thus

$$\|u_k(t)\|_2 \le C\,2^{ka'}\,\lambda_{k-1}^2\,.$$

Where $a'>0$, therefore

$$\ln\|u_k(t)\|_2 \le \ln\ell_K \le \ln C+k\ln B+2\ln \mathrm{I}_{K-1}$$

Where $B=2^{a'}>1$, therefor

$$\ln\|u_k(t)\|_2 \le \ln C\sum_{i=0}^{k-2}2^i-2^{k-1}\ln\lambda_1+\ln B(\sum_{j=0}^{k-2}(k-j)2^j)$$

$$\le(2^{k-1}-1)\ln C+2^{k-1}\ln\lambda_1+f(k)\ln B,$$ with

$f(k)=2^{k+1}-2^{k-1}-k-2$, that is,

$$\|u_k(t)\|_{m_k+2} \le \left\{C^{2^{k-1}}\lambda_1^{2^{k-1}}B^{f(k)}\right\}^{\frac{2}{p_k+2}}$$

Let $k\to\infty$, to get

$\|u_k(t)\|_\infty \leq C\lambda_1^2 \leq C(\max\{1, sup_t\|u(t)\|^2\})^2$ (16)

From (10), and $m_k = 0$, to get

$\frac{d^2}{dt^2}\|u(t)\|_2^2 + (C_1 - 1)\|\nabla u(t)\|^2 \leq 2\, C_2(\|u(t)\|_2^2 + C\,\|u(t)\|^3 + \|u(t)\|)$

According to Poincare's inequality, to have

$C_p\,\|u(t)\|_2^2 \leq \|\nabla\, u(t)\|_2^2$

So, $|\Omega|$ is sufficiently small, to have $(C_1 - 1)\, C_p < 2C_2$, then by young's inequality, get

$\frac{d^2}{dt^2}\|u(t)\|_2^2 + (C_1 - 1)\, C_p\,\|u(t)\|^2 \leq 2\, C_2(\|u(t)\|^2 + C\,\|u(t)\|^3 + \|u(t)\|)$

Let $2\, C_2 > C$ and $2\, C_2 > 1$, then

$\frac{d^2}{dt^2}\|u(t)\|_2^2 \leq 2\, C_2(\|u(t)\|^2 + \|u(t)\|^3 + \|u(t)\|)$

Thus,

$\frac{d^2}{dt^2}\|u(t)\|_2^2 \leq C\|u(t)\|_2^2$, therefore $\frac{d}{dt}\|\acute{u}(t)\|_2^2 \leq C\|u(t)\|_2^2$

To obtain $\|u(t)\|_{L^\infty(Q_T)} < \breve{R}$ where $\breve{R}=|\Omega|R$.

To prove the following results, the following auxiliary problem has been need:

$$\begin{cases} \dfrac{\partial^2 u_\epsilon}{\partial t^2} + D_i\left((1-\tau)\gamma D_i u_\epsilon + \tau a_{ij}(x,t,u)D_j u_\epsilon\right) + \epsilon u_\epsilon - g(x,t,u_\epsilon) = f(x,t) & (17) \\ \dfrac{\partial u_\epsilon}{\partial n} = 0, \quad (n,t) \in \partial\Omega \times (0,T) & (18) \\ u_\epsilon(x,0) = u_{\epsilon_1}(x), \quad x \in \Omega & (19) \\ u_\epsilon{}'(x,0) = u_\epsilon{}'(x,T)\,, \quad x \in \Omega. & (20) \end{cases}$$

where $Q_T = \Omega \times (0,\mathrm{T})$, $u_{1\epsilon} \in C^\infty(\Omega)$ satisfy

$0 \leq u_1 \leq \|u_1\|_{L^\infty(\Omega)}$, $W^{1,2}(\Omega)\; as\; \epsilon \to 0$ in $u_{1\epsilon} \to u_1$, (21)

where $\tau \in [0,1]$.

The standard parabolic theory [10] shows that (17)-(19) give a nonnegative classical solutionu_ϵ, where the solution of Eq. (1)-(4) is a limit point of $u_\epsilon$of Eq. (17) and (19). As a result of the same results proving in [5], to have the results.

Corollary (2.1):

Consider Eq. (17) - (19). Then ∃a positive constant R independent of the ϵ and

$\ni$ $\deg(I - G(1, g(x, u_\epsilon) + f(x,t), B_R, 0) = 1$

Where B_R is a ball centered at the origin with radius R in $L^\infty(Q_T)$.

Lemma (2.1):

Consider the problem (17)-(19).Then ∃ constants $r_0 > 0$ and

$\epsilon > 0$ ∋for any $0 \le r < r_0$, $\epsilon < \epsilon_0$, $u = G(\tau, g(x, u_\epsilon) + f(x,t) + (1-\tau))$, admits no nontrivial solution u satisfying $0 \le \|u\|_{L^\infty(Q_T).} \le r$, where r independent of ϵ.

Corollary (2.2):

Consider the problem (17)-(19).Then ∃ a small constant $0 < r < R$ which is independent of ϵ, τ ∋$\deg(1 - G(1, (g(x, u_\epsilon) + f(x,t)), B_r, 0) = 0$, where B_r is a ball in $L^\infty(Q_T)$.

Now, in this paper, to give the proof of the result have $\deg\left(1 - G(f(.)), \frac{B_R}{B_r}, 0\right) = 1$ is the proof of theorem 1 using corollaries 1, and 2, to have, $R > r > 0$. Eq. (17)-(19) admits a nonnegative nontrivial solution u_ϵ with $r \le \|u_\epsilon\|_\infty \le |\Omega|R$. This can be demonstrated by combining the regularity results [11] and a similar argument as in [12].

3. Conclusions

The proposal equations (1)-(4) with their admissible conditions suitable of their components have been concluded as different technical and complexity than parabolic and elliptic differential equations, and the existence of the solution is dependent on first and second derivatives, as well as some of the special spaces that appear in the construction of the proposal equation to guaranty the necessary properties to be the solution is satisfied.

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الحل الدوري لمسألة القطع الزائد التفاضلية بشروط اولية لدالة السيطرة

زينب عبد الامير خضر , سمير قاسم حسن

قسم الرياضيات ,كلية التربية ,الجامعة المستنصرية

المستخلص

تم دراسة الحل والسيطرة المثلى لصنف من المعادلات التي من نوع القطع الزائد ذات حل دوري في توفر شروط تحققت باسلوب نظرية درجة لاري شويدر مع وجود دالة سيطرة كمؤثر على المعادلة او النظام من حيث انها تكون دالة سيطرة كدالة قيمة حدودية غير محلية. كذلك النظام ذات القطع الزائد المعرف على مجال مقيد والتي تحوي على جزء التحميل ومصدر بسيط كدالة غير خطية. وان الصنف المعروض حق وجدانية ووحدانية الحل بالاعتماد على شروط ضرورية ومتباينات معقدة لضمان الحل.