Isolated Singularities of Positive Harmonic Functions in Bounded Domains

Haneen R. Hamad and Khaldoun S. Al-Yasiri*

Department of Mathematics, College of Science, University of Basrah, Basra, Iraq

*Corresponding author E-mail: Khaldoun.ghalib@uobasrah.edu.iq

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ABSTRACT

The singularity of positive harmonic functions has been studied in the punctured unit open ball in $B(0,1)\setminus\{0\}$ [1]. The representation of a positive harmonic function in $B(0,1)\setminus\{0\}$ is given by Bôcher's Theorem. In this work, the representation of Bôchre's has been generalized to a punctured bounded domain in R^n, $\Omega\setminus\{x_0\}$. Then the work has been extended to represent positive harmonic functions with two isolated singularities in a bounded domain of R^n.

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1. Introduction

This work includes study the isolated singularities of harmonic functions and then it has been described the features of positive harmonic functions near the isolated singularities in a bounded domain of R^n. Harmonic functions are the main part of the field of elliptic partial differential equations [2]. Studying the isolated singularities of harmonic functions would lead to studying the singularities of elliptic partial differential equations. Partial differential equations (PDEs) are one of the most desirable topics in mathematics that plays an important role in many scientific fields such as physics, engineering, astronomy, and medicine. Recently, many researchers have been focusing on using problems that arise from partial differential equations to solve them in numerical study. For example, the stability of the heat pattern in porous media has been studied in [3-5]. The numerical analysis of Newtonian and die swell flow have been studied in [6,7]. Inpainting techniques and topological analysis have been used to develop the missing details of images with involving PDEs can be found in [8]. For the domain decomposition based on system of partial differential equations that have been treated in numerical programs have been shown in [9] and [10]. Another type of work for the researchers that have used functional analysis and calculus of variation to deal with elliptic partial differential equations can be seen in [11], and [12].

Laplace equation is the core part of the elliptic partial differential equations which can be considered as the concrete base of all theorems that treated the elliptic partial differential equations. In physics and in dimension three, Laplace equation can be derived from Maxwell's equations. For more information about the derivation see [13]. The purpose of this work is to generalize the Bôcher's representation of positive harmonic functions in the punctured unit open ball. In fact, in [1], any positive harmonic function in $B(0,1)\backslash\{0\}$ has the following form

$$u(x)=\begin{cases}\tilde{A}|x|^{2-n}+\tilde{u}(x), & n\geq 3;\\ \tilde{A}\log\left(\frac{1}{|x|}\right)+\tilde{u}(x), & n=2,\end{cases} \tag{1}$$

where, $\tilde{A}$ is a nonnegative constant and $\tilde{u}$ is a harmonic function in $B(0,1)$.

First, Bôcher's representation has extended into a general punctured bounded domain in R^n. That is, when Ω is a bounded domain in R^n, and u is a positive harmonic in $\Omega\backslash\{x_0\}$. The following representation for u in $\Omega\backslash\{x_0\}$ has been obtained,

$$u(x)=\begin{cases}\tilde{A}\log\left(\frac{\alpha}{|x-x_0|}\right)+v(x), & \text{if} \quad n=2;\\ \tilde{A}|x-x_0|^{2-n}+v(x), & \text{if} \quad n\geq 3.\end{cases} \tag{2}$$

Where, $\tilde{A}$ is a nonnegative constant, v is a harmonic function in Ω, and

$$\alpha=\begin{cases}1 & \text{if} \quad \operatorname{dist}(x_0,\partial\Omega)>1;\\ 1-\epsilon & \text{if} \quad \operatorname{dist}(x_0,\partial\Omega)=1;\\ \operatorname{dist}(x_0,\partial\Omega) & \text{if} \quad \operatorname{dist}(x_0,\partial\Omega)<1,\end{cases} \tag{3}$$

for some $0<\epsilon<1$.

Moreover, the representation of positive harmonic functions extended into bounded domains with two singularities. In other words, for a positive harmonic function u in $\Omega\backslash\{a_1,a_2\}$, where a_1 and a_2 be two isolated singularities of u in the bounded domain Ω, where the closed neighborhood for a_1, $\overline{B(a_1,\delta_1)}\subset\Omega$, and the closed neighborhood for a_2, $\overline{B(a_2,\delta_2)}\subset\Omega$ are disjoint, that is,

$$\overline{B(a_1,\delta_1)}\cap\overline{B(a_2,\delta_2)}=\emptyset.$$

The following representation has been obtained

$$u(x)=\begin{cases}A_1\log\left(\frac{\alpha_1}{|x-a_1|}\right)+A_2\log\left(\frac{\alpha_2}{|x-a_2|}\right)+v(x) & \text{if} \quad n=2,\\ A_1\,|x-a_1|^{2-n}+A_2\,|x-a_2|^{2-n}+v(x) & \text{if} \quad n\geq 3,\end{cases} \tag{4}$$

for A_1, A_2 to be nonnegative constants, v is harmonic function in Ω,

$$\alpha_1=\begin{cases}1 & \text{if} \quad \delta_1>1;\\ 1-\epsilon & \text{if} \quad \delta_1=1;\\ \delta_1 & \text{if} \quad \delta_1<1;\end{cases} \tag{5}$$

and,

$$\alpha_2 = \begin{cases} 1 & \text{if } \delta_2 > 1, \\ 1-\epsilon & \text{if } \delta_2 = 1, \\ \delta_2 & \text{if } \delta_2 < 1, \end{cases} \quad (6)$$

for some $0 < \epsilon < 1$.

This paper is organized as follows: Section (2) includes all ingredients that are needed to clarify the notion of the isolated singularities and the radial average of harmonic functions in a punctured open unit ball. Furthermore, Bôcher's representation for the positive harmonic function in the punctured open unit ball in R^nhas been stated. Section (3) generalizes the representation of positive harmonic functions into a general punctured bounded domain of R^n instead of the punctured open unit ball. Continuing further in section (4) for generalizing the representation of harmonic functions with two singularities in a bounded domain of R^n. In section (5), the conclusions of this work have been stated. The future work is stated in section (6).

1. Preliminaries

In this section includes some basic details about the harmonic functions and the singularities of positive harmonic functions in a bounded domain of R^n. The representation of Bôcher's that has been studied in [1] will be briefly stated in this section.

2.1. Harmonic Functions

Let Ω be an open set of R^n, where $n \geq 1$. Denote by $x = (x_1, x_2, \dots, x_n)$ to be a point in Ω, and $\frac{\partial^j}{\partial x_i^j}$ to be the jth partial derivative with respect to x_i coordinate for $i = 1,2,\dots,n$ and $j = 1,2$. The boundary of Ω is denoted by $\partial\Omega$ and the distance from x to the boundary $\partial\Omega$ is given by the following equation

$$\mathrm{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|.$$

Definition 2.1. [2] The Laplacian operator on R^n is defined by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \quad (7)$$

Definition 2.2. [2] A C^2 −function u that is defined on Ω is called harmonic on Ω if it satisfies the Laplace's equation

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \quad (8)$$

for all $x \in \Omega$.

A C^2 −function means that all of its second partial derivatives are exist and continuous in its domain of definition.

Definition 2.3. [14] The *fundamental solution for Laplacian* Δ is given by the following equation

$$\Gamma(x) = \begin{cases} \frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}} , & \text{for} \quad n \geq 3 , \\ -\frac{1}{2\pi} \log|x| , & \text{for} \quad n = 2, \end{cases} \quad (9)$$

where ω_n is the volume of the unit open ball $B(0,1)$ in R^n .

The symbol $|.|$ denotes for the magnitude (norm) of the point $x = (x_1, x_2, \dots, x_n)$ in R^{n}, that is, $|x| = \sqrt{\mathrm{x}_1^2 + \cdots + \mathrm{x}_\mathrm{n}^2}$.

The ***punctured unit open*** ball is the unit open ball in R^n excluding its center, and it is denoted by $B(1)$. That is,

$$B(1) = B(0,1)\backslash\{0\}. \quad (10)$$

2.2 Singularities of Harmonic Functions

This subsection is devoted for stating basic details about the singularities of harmonic functions.

Let Ω be abounded domain in R^n and u be a given harmonic function. The following useful information about the singularities are given;

1. If u is defined in Ω except at a point $\tilde{x} \in \Omega$. In this case, the point $\tilde{x}$ is called a ***singular*** point for u in the bounded domain Ω.

2. If $\tilde{x}$ is a singularity of u and if there exists $r > 0$ such that $B(\tilde{x}, r) \subset \Omega$ and $\tilde{x}$ is the only singularity of u in $B(\tilde{x}, r)$. In this case, $\tilde{x}$ is called an ***isolated singularity*** of u in Ω.
3. If $\tilde{x}$ is an isolated singularity of u in Ω and if there exists another harmonic function $\tilde{u}$ which is defined in Ω so that

$$\tilde{u}(x) = u(x), \quad x \in \Omega \backslash \{\tilde{x}\},$$

in this case $\tilde{x}$ is called a ***removable isolated singularity*** of u and $\tilde{u}$ is called

harmonic extension of u in Ω.

4. Let u be a harmonic function defined in the bounded domain Ω. For a positive constant α, ***the dilation of u*** is defined to be the harmonic function $\tilde{u}_\alpha$ that defined in the bounded domain $\Omega_{\frac{1}{\alpha}}$ as follows,

$$\tilde{u}_\alpha(x) = u(\alpha x), \; x \in \Omega_{\frac{1}{\alpha}} = \left\{\frac{x}{\alpha} : \; x \in \Omega\right\}. \tag{11}$$

Let $\tilde{x}$ be an isolated singularity of u in Ω. Since the translation of harmonic function is harmonic, then the following function $v(x) = u(x + \tilde{x})$, is harmonic in $\tilde{\Omega} = \Omega - \{\tilde{x}\} = \{x - \tilde{x} : x \in \Omega\}$. Therefore, if $\tilde{x}$ is an isolated singularity of u in Ω, then 0 is an isolated singularity for v in $\tilde{\Omega}$. Thus, without loss of generality, instead of saying that $\tilde{x}$ is a singularity of u, after a suitable translation, Then, 0 is a singularity of u in Ω.
5. If $\tilde{x}$ is a singularity of u in Ω, then it can be assumed that, after a suitable translation and dilation, Ω is containing the closed unit ball $\overline{B(0,1)}$ and 0 is the singular point of u instead of $\tilde{x}$.

2.3. Radial Average of Harmonic Functions

Let u be a harmonic function defined in $B(0,1)\backslash\{0\}$. The ***radial average (spherical average)*** of u in the punctured unit open ball [1], and [15] , $0 < r = |x| < 1$, is defined as follows

$$R_u(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(rw) ds(w). \tag{12}$$

Proposition 2.4. [1] The radial average of u in $B(0,1)\backslash\{0\}$ satisfying the following equation

$$R_u(x) = \begin{cases} C_1 + C_2|x|^{2-n}, & n \geq 3, \\ C_1 + C_2 \log\left(\frac{1}{|x|}\right), & n = 2, \end{cases} \tag{13}$$

for some constants C_1 and C_2.

2.3. Positive Harmonic Functions in the Punctured Unit Open Ball

This subsection, briefly summarizes the representation of positive harmonic functions in $B(0,1)\backslash\{0\}$ that has been discussed in [1] and it will show that such functions can be decomposed into two parts. One part depends on the fundamental solution of Laplacian and the other part is harmonic in the entire unit ball. In other words, if u is positive harmonic in $B(0,1)\backslash\{0\}$, then u has the following decomposition in $B(0,1)\backslash\{0\}$,

$$u(x) = \begin{cases} \tilde{A}|x|^{2-n} + \tilde{u}(x), & n \geq 3, \\ \tilde{A} \log\left(\frac{1}{|x|}\right) + \tilde{u}(x), & n = 2, \end{cases} \tag{14}$$

where, $\tilde{u}$ is a harmonic function in $B(0,1)$ and $\tilde{A}$ is a nonnegative constant.

The above representation of positive harmonic functions is called Bôcher's representation and it has been studied in [1].

Remark 2.5. It can be shown that the harmonic function $\tilde{u}$ and the nonnegative constant $\tilde{A}$ in the representation that is given by (14) are unique. In fact, let us assume that u has two representations. That is, there are two nonnegative constants A_1, A_2 and two harmonic functions, u_2, u_2 in $B(0,1)$, the work designated for $n \geq 3$ (similar work would simply hold for $n = 2$, be such that

$$u(x) = A_1|x|^{2-n} + u_1(x), \qquad x \in B(0,1)\backslash\{0\},$$

and,

$$u(x) = A_2|x|^{2-n} + u_2(x), \quad x \in B(0,1)\backslash\{0\}.$$

Therefore, it obtained that

$$u_1(x) - u_2(x) = (A_2 - A_1)\,|x|^{2-n},\ \forall\, x \in B(0,1)\backslash\{0\}. \tag{15}$$

Involving the Laplace operator Δ for both sides of (15), obtaining the following differential identity,

$$\Delta(u_1 - u_2)(x) = (A_2 - A_1)\,\Delta|x|^{2-n} \tag{16}$$

Therefore, the following identity is abstained

$$(A_2 - A_1)\delta_0(x) = 0, \qquad x \in B(0,1). \tag{17}$$

Where δ_0 is the Dirac delta function concentrating at $x = 0$, which is defined by the following identity,

$$\delta_0(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ +\infty & \text{if } x = 0. \end{cases} \tag{18}$$

Therefore, it is seen that $A_1 = A_2$ and $u_1 = u_2$. That is, the Bôcher's representation in (14) is unique.

2. Positive Harmonic Functions in the Punctured Bounded Domain

The aim of this section is to generalize the representation of positive harmonic functions into general punctured bounded domain instead of punctured unit open ball.

Let Ω be a bounded domain in R^n and u be a positive harmonic function in $\Omega\backslash\{x_0\}$, where x_0 is an isolated singularity of u in Ω.

First, involve the translation

$$u^*(x) = u(x + x_0),\ \text{for } x \in \Omega^*\backslash\{0\},$$

where,

$$\Omega^* = (\Omega - \{x_0\}) = \{y - x_0 : y \in \Omega\}.$$

Therefore u^* is positive harmonic function in $\Omega^*\backslash\{0\}$.

After translating u into u^*, dilate u^* to u^*_α in the domain $\Omega^*_{\frac{1}{\alpha}}$ in order to guarantee that $B(0,1) \subset \Omega^*_{\frac{1}{\alpha}}$. In fact, the desired dilation can be represented as follows;

For $d = \mathrm{dist}(0, \partial\Omega^*)$, make the following two cases

1. If $d > 1$, set $\alpha = 1$ and define

$$\Omega^*_{\frac{1}{\alpha}} = \Omega^* \text{ and } u^*_\alpha(x) = u(x) \text{ for } x \in \Omega^*_{\frac{1}{\alpha}} . \tag{19}$$

Then evidently it can be seen the inclusion

$$B(0,1) \subset \Omega^*_{\frac{1}{\alpha}} .$$

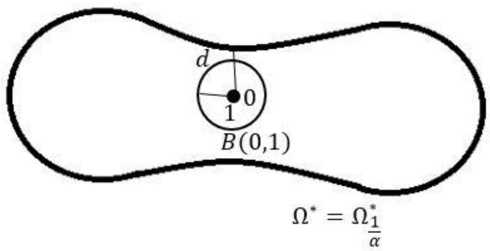

Figure 1: The dilated domain for d>1.

2. If $d \le 1$, set

$$\alpha = \begin{cases} d & \text{if } d < 1; \\ 1-\epsilon & \text{if } d = 1; \end{cases} \tag{20}$$

for some $0 < \epsilon < 1$. Then define the following dilation

$$u^*_\alpha(x) = u^*(\alpha x), x \in \Omega^*_{\frac{1}{\alpha}} ,$$

where,

$$\Omega^*_{\frac{1}{\alpha}} = \left\{\frac{y}{\alpha} : y \in \Omega^*\right\}.$$

Thus, it obtains the desired inclusion,

$$B(0,1) \subset \Omega^*_{\frac{1}{\alpha}}.$$

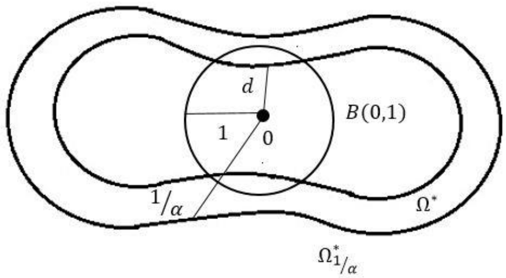

Figure 2: The dilated domain for d≤ **1.**

From the above two cases, it has obtained that u_α^* is positive harmonic in $B(0,1)\backslash\{0\}$. Therefore, from the previous section (4.4), the following representation for u_α^* in $B(0,1)\backslash\{0\}$ is obtained,

$$u_\alpha^*(x) = \begin{cases} A\log\left(\frac{1}{|x|}\right) + v^*(x), & n = 2; \\ A|x|^{2-n} + v^*(x), & n \geq 3; \end{cases} \tag{21}$$

for unique $A \geq 0$ and v^* is the unique harmonic function in $B(0,1)$.

Extend v^* to be harmonic in $\Omega^*_{\frac{1}{\alpha}}$ as follows

$$\overline{v^*}(x) = \begin{cases} v^*(x) & \text{if} \quad x \in B(0,1); \\ u_\alpha^*(x) - A\log\left(\frac{1}{|x|}\right) & \text{for } n = 2 \text{ and } x \in \Omega^*_{\frac{1}{\alpha}}\backslash B(0,1); \\ u_\alpha^*(x) - A|x|^{2-n} & \text{for } n \geq 3 \text{ and } x \in \Omega^*_{\frac{1}{\alpha}}\backslash B(0,1). \end{cases} \tag{22}$$

Thus $\overline{v^*}$ is harmonic function in $\Omega^*_{\frac{1}{\alpha}}$.

Therefore,

$$u_\alpha^*(x) = \begin{cases} A\log\left(\frac{1}{|x|}\right) + \overline{v^*}(x) & \text{if } n = 2; \\ A|x|^{2-n} + \overline{v^*}(x) & \text{if } n \geq 3; \end{cases} \tag{23}$$

for $x \in \Omega^*_{\frac{1}{\alpha}}\backslash\{0\}$.

Now, dilate back to obtain the following,

$$u^*(x) = u_\alpha^*\left(\frac{x}{\alpha}\right), \text{ for } x \in \Omega^*\backslash\{0\}.$$

That is, for $x \in \Omega^*\backslash\{0\}$, it follows that

$$u^*(x) = \begin{cases} A\log\left(\frac{\alpha}{|x|}\right) + \overline{v^*}\left(\frac{x}{\alpha}\right), & n = 2; \\ (\alpha^{n-2}A)|x|^{2-n} + \overline{v^*}\left(\frac{x}{\alpha}\right), & n \geq 3. \end{cases} \quad (24)$$

Let

$$\tilde{A} = \begin{cases} A & \text{if } \ n = 2; \\ \alpha^{n-2}A & \text{if } \ n \geq 3; \end{cases} \quad (25)$$

and $\tilde{v}(x) = \overline{v^*}\left(\frac{x}{\alpha}\right)$, for $x \in \Omega^*$, then for $x \in \Omega^*\backslash\{0\}$ it follows that

$$u^*(x) = \begin{cases} \tilde{A}\log\left(\frac{\alpha}{|x|}\right) + \tilde{v}(x), & \text{if } \ n = 2; \\ \tilde{A}\,|x|^{2-n} + \tilde{v}(x), & \text{if } \ n \geq 3; \end{cases} \quad (26)$$

where $\tilde{A} \geq 0$ and $\tilde{v}$ is harmonic in Ω^*.

Finally, translate back as follows:

$$u(x) = u^*(x - x_0) \text{ for } x \in \Omega\backslash\{x_0\}.$$

Therefore, for $x \in \Omega\backslash\{x_0\}$, it follows that

$$u(x) = \begin{cases} \tilde{A}\log\left(\frac{\alpha}{|x-x_0|}\right) + v(x), & \text{if } \ n = 2, \\ \tilde{A}|x - x_0|^{2-n} + v(x), & \text{if } \ n \geq 3, \end{cases} \quad (27)$$

where, $v(x) = \tilde{v}(x - x_0)$ is a harmonic function in Ω, and for some $0 < \epsilon < 1$,

$$\alpha = \begin{cases} 1, & \text{if } \ d > 1, \\ 1 - \epsilon, & \text{if } \ d = 1, \\ d, & \text{if } \ d < 1. \end{cases} \quad (28)$$

Remark 3.1. If $\tilde{A} = 0$ in (27), then x_0 is a removable isolated singularity of u in Ω.

4. Positive Harmonic Functions in Bounded Domain with more than One Singularity

In the previous section, the representation of positive harmonic functions in a general punctured bounded domain is given. Continuing further to generalize the representation of positive harmonic functions in a bounded domain that contains more than one isolated singularity. For this purpose, the following definition is needed to ease the representation of such positive harmonic functions with two isolated singularities.

Definition 4.1. Let S be a subset of R^n, the characteristic function of S, χ_S, is defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if} \quad x \in S; \\ 0 & \text{if} \quad x \notin S. \end{cases} \tag{29}$$

The work will focus on bounded domains with two isolated singularities because positive harmonic functions in bounded domains with more than two isolated singularities can be made in the same way.

Let u be a positive harmonic function in $\Omega\backslash\{a_1, a_2\}$, where a_1 and a_2 be two isolated singularities of u in the bounded domain Ω.

First, let $\overline{B(a_1, \delta_1)} \subset \Omega$ and $\overline{B(a_2, \delta_2)} \subset \Omega$ be closed neighborhoods for a_1 and a_2 respectively be such that $\overline{B(a_1, \delta_1)} \cap \overline{B(a_2, \delta_2)} = \emptyset$.

From the representation in the previous section in $B(a_1, \delta_1)$ it follows that

$$u(x) = \begin{cases} A_1 \log\left(\frac{\alpha_1}{|x-a_1|}\right) + v_1(x) & \text{if} \quad n = 2; \\ A_1 |x - a_1|^{2-n} + v_1(x) & \text{if} \quad n \geq 3; \end{cases} \tag{30}$$

where $A_1 \geq 0$ and v_1 is harmonic in $B(a_1, \delta_1)$, and

$$\alpha_1 = \begin{cases} 1 & \text{if} \quad \delta_1 > 1; \\ 1 - \epsilon & \text{if} \quad \delta_1 = 1; \\ \delta_1 & \text{if} \quad \delta_1 < 1; \end{cases} \tag{31}$$

for some $0 < \epsilon < 1$.

Notice that, the harmonic function v_1 can be extended harmonically to $\partial B(a_1, \delta_1)$ as follows, for $y \in \partial B(a_1, \delta_1)$, define $v(y) = \lim_{x \to y} v(x)$.

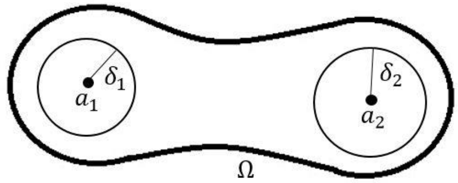

Figure 3: Bounded domain with two isolated singularities

Therefore, in $\overline{B(a_1,\delta_1)}\backslash\{a_1\}$, u has the following representation;

$$u(x) = \begin{cases} A_1 \log\left(\frac{\alpha_1}{|x-a_1|}\right) + v_1(x) & \text{if } n = 2; \\ A_1|x-a_1|^{2-n} + v_1(x) & \text{if } n \geq 3; \end{cases} \tag{32}$$

also, in $\overline{B(a_2,\delta_2)}$, u can be represented as follows;

$$u(x) = \begin{cases} A_2 \log\left(\frac{\alpha_2}{|x-a_2|}\right) + v_2(x) & \text{if } n = 2, \\ A_2\,|x-a_2|^{2-n} + v_2(x) & \text{if } n \geq 3, \end{cases} \tag{33}$$

where, $A_2 \geq 0$, v_2 is harmonic in $\overline{B(a_2,\delta_2)}$, and

$$\alpha_2 = \begin{cases} 1 & \text{if } \delta_2 > 1, \\ 1-\epsilon & \text{if } \delta_2 = 1, \\ \delta_2 & \text{if } \delta_2 < 1, \end{cases} \tag{34}$$

for some $0 < \epsilon < 1$.

Now use v_1 and v_2 to define the harmonic function v in Ω depending on the dimension n. for $n = 2$, define

$$\begin{aligned} v(x) &= \chi_{B(a_1,\delta_1)}\left(v_1(x) - A_2\log\left(\frac{\alpha_2}{|x-a_2|}\right)\right) + \\ &\chi_{B(a_2,\delta_2)}\left(v_2(x) - A_1\log\left(\frac{\alpha_1}{|x-a_1|}\right)\right) + \\ &\chi_{\Omega\backslash(B(a_1,\delta_1)\cup B(a_2,\delta_2))}\left(\mathrm{u}(x) - A_1\log\left(\frac{\alpha_1}{|x-a_1|}\right) - A_2\log\left(\frac{\alpha_2}{|x-a_2|}\right)\right). \end{aligned} \tag{35.a}$$

While for $n \geq 3$ the function v is given by the form

$$v(x) = \chi_{B(a_1,\delta_1)}\left(v_1(x) - A_2|x - a_2|^{2-n}\right) + \chi_{B(a_2,\delta_2)}\left(v_2(x) - A_1|x - a_1|^{2-n}\right) + \chi_{\Omega\setminus(B(a_1,\delta_1)\cup B(a_2,\delta_2))}\left(u(x) - A_1|x - a_1|^{2-n} - A_2|x - a_2|^{2-n}\right). \quad (35.b)$$

Therefore, v is harmonic function in Ω. Consequently, u have the following representation in $\Omega\setminus\{a_1, a_2\}$

$$u(x) = \begin{cases} A_1 \log\left(\frac{\alpha_1}{|x-a_1|}\right) + A_2 \log\left(\frac{\alpha_2}{|x-a_2|}\right) + v(x) & \text{if } \ n = 2, \\ A_1 |x - a_1|^{2-n} + A_2 |x - a_2|^{2-n} + v(x) & \text{if } \ n \geq 3. \end{cases} \quad (36)$$

6. Conclusions

The positive harmonic function u in $\Omega\setminus\{x_0\}$, where Ω is a bounded domain of R^n and x_0 is an isolated singularity of u in Ω can be divided into two parts. One of them is the singular parts that depends on the fundamental solution of Laplacian and the other part is the harmonic part. Explicitly, u has given in (27). The work has continued to describe more general form for positive harmonic functions with two isolated singularities. That is, when u is a positive harmonic function in $\Omega\setminus\{a_1, a_2\}$, for two isolated singularities a_1 and a_2 of u in the bounded domain Ω. Insert the condition $\overline{B(a_1,\delta_1)} \cap \overline{B(a_2,\delta_2)} = \emptyset$ and $\overline{B(a_1,\delta_1)} \subset \Omega$, $\overline{B(a_2,\delta_2)} \subset \Omega$, would enable to generalize the formula (27) for two isolated singularities. In which case, u described in (36).

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النقاط المعتلة للدوال التوافقية الموجبة في المجال المقيد

حنين رياض حمد ، خلدون سعد غالب

قسم الرياضيات، كلية العلوم، جامعة البصرة

المستخلص

لقد تم دراسة النقطة المعتلة للدوال التوافقية الموجبة في الكرة التي يكون مركزها هو النقطة المعتلة من قبل الباحثون ألكس و بوردن و رأيمي كما مؤشر في المصدر الاول حيث ان تمثيل هذه الدالة يكون بواسطة مبرهنة بوجر. لقد تم تعميم الدراسة الى مجال مقيد عام يحتوي على نقطة معتلة واحدة بدلاً من الكرة التي يكون مركزها هو النقطة المعتلة. ومن ثم تم تعميم الدراسة الى اعطاء صيغة الدوال التوافقية في مجال مقيد عام يحتوي على نقطتين معتلتين.