Modify Lyapunov-Schmidt Method for Nonhomogeneous Differential Equation

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ABSTRACT

The Lyapunov-Schmidt reduction for nonhomogeneous problems is modified when the dimension of the null space is two. The novel method was utilized to approximate the solutions of the nonlinear wave equation. This equation's related key function has been identified.

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1. Introduction

Many phenomena with nonlinear behavior occur in physics and mathematics, and they may be represented using the nonlinear Fredholm operator provided by:

$$f(x,\gamma) = b, x \in S \subseteq X, b \in Y, \gamma \in R^n \quad . \tag{1}$$

where f is a smooth Fredholm map with zero index, $X\ and\ Y$ are real Banach spaces and $S \subseteq X$ is open. The method of reduction to dimensional equation can be used to solve this equation as below:

$$\theta(\xi,\gamma) = \beta, \xi \in M, \beta \in N, \tag{2}$$

Where $M\ and\ N$ are smooth manifolds with finite dimension. However, Eq 1 can be reduced to Eq2 by Lyapunov-Schmidt method where Eq2 has all the topological (multiplicity) and analytical (bifurcation diagram) properties [1,2]. Lyapunov-Schmidt method (LSM) can be used for a variety of reasons such that the solutions of unlimited dimensional spaces that coincide with the solutions of limited dimensional spaces as well as to obtain the bifurcation solutions of nonlinear partial differential equations that appear in mathematics. Therefore, LSM becames an important method in the modern mathematics. Krasnoselskii (1956) [3] was used this method but with the name of alternative method because of the method was applied when the implicit function theory cannot use to study bifurcation of extremely in the classical case (without boundaries). Y.I Sapronov [1] are employed the complement solution to get the linear Ritz approximation of the function in Eq. 1 that represented by the function $\mathcal{W}(\zeta,\lambda)$. The bounded value problems using the Lyapunov-Schmidt method (LSM) was studied by many researchers [5,6]. Abdul Hussain and Mizeal [6] has been studied the bifurcation of differential equation with the boundary conditions and found that the elastic beams equation has a bifurcation equation is given by a nonlinear system of two algebraic equations. Furthermore, Abdul Hussain [7] introduces a modify method to find the nonlinear Ritz approximation of Fredholm functional through studding the following problems:

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0, \tag{3}$$

And with boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0 \tag{4}$$

Found that the nonlinear Ritz approximation is a function given by:

$$W_3(\xi,\delta) = {\xi_1}^{12} + {\xi_2}^{12} + \lambda_1{\xi_1}^2{\xi_2}^{10} + \lambda_2{\xi_1}^4{\xi_2}^8 + \lambda_3{\xi_1}^6{\xi_2}^6 + \lambda_4{\xi_1}^8{\xi_2}^4 + \lambda_5{\xi_1}^{10}{\xi_2}^2$$
$$+ \lambda_6{\xi_1}^2{\xi_2}^8 + \lambda_7{\xi_1}^8{\xi_2}^2 + \lambda_8{\xi_1}^6{\xi_2}^4 + \lambda_9{\xi_1}^4{\xi_2}^6 + \lambda_{10}{\xi_1}^8$$
$$+ \lambda_{11}{\xi_2}^{11} + \lambda_{12}{\xi_1}^6{\xi_2}^2 + \lambda_{13}{\xi_1}^2{\xi_2}^6 + \lambda_{14}{\xi_1}^4{\xi_2}^4$$
$$+ \lambda_{15}{\xi_1}^6 + \lambda_{16}{\xi_2}^6 + \lambda_{17}{\xi_1}^4{\xi_2}^2 + \lambda_{18}{\xi_1}^2{\xi_2}^4 + \lambda_{19}{\xi_1}^4 + \lambda_{20}{\xi_2}^4 + \lambda_{21}{\xi_1}^2{\xi_2}^2$$
$$+ \lambda_{22}{\xi_1}^2 + \lambda_{23}{\xi_2}^2 + O(|\zeta|^{12}) + O(|\zeta|^{12})O(|\delta|) \quad (5)$$

where $\xi = (\xi_1, \xi_2)$ and $\delta = \{\lambda_i\}_{i=1}^{23}$ where λ is a parameter.

Murtada [8] was used Lyapunov-Schmidt reduction (LSR) to study the bifurcation solutions and bifurcation diagram of the following boundary value problem:

$$\frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + \beta w + ww'' = \psi \quad (6)$$

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0 \quad (7)$$

Recently, Lyapunov-Schmidt reduction (LSR) was used to find bifurcation solution of the boundary condition problem where the solution of extremes of the functions of codimensions eight and five at the origin. Z.A Shawi and M.A Abdul Hussain [9] are used the local method of Lyapunov-Schmidt to find the bifurcation solutions of non-linear differential equations of the fourth order in three parameters while H.K. Kadhim and M.A Abdul Hussain [10] were studied the bifurcation solution of extremes of the functions of codimensions eight and five at the origin by using Lyapunov-Schmidt reduction (LSR). Recently, critical points of functions with four variables and eight parameters are classified [11]. In previous works, the presence and absence of u shaped solutions were studied using the Lyapunov-Schmidt method and Ritz linear approximation. In the present work, the presence and absence of $u + v$ solutions using the modify Lyapunov-Schmidt method and the nonlinear Ritz approximation are studied.

2. Lyapunov-Schmidt reduction (LSR)

Schmidt in 1908 was suggested a method to obtain the solutions of operator equation and in particular to solve the problems that possess variational property and those that unpossessed variational property. The method gives as follows:

Suppose that E and M are real Banach spaces and $G: E \to M$ is a nonlinear Fredholm operator of index zero and G is given by

$$G(z, \lambda) = 0, \;\; z \in E, \;\; \lambda \in \mathbb{R}^n. \tag{8}$$

The spaces E and M can be written as the following direct sum:

$$E = N \oplus N^{\perp}, \tag{9}$$

$$M = \tilde{N} \oplus \tilde{N}^{\perp} \tag{10}$$

where N and $\tilde{N}$ are n-dimensional subspaces of E and M, respectively while $N^{\perp}$and $\tilde{N}^{\perp}$ are orthogonal spaces of N and $\tilde{N}$ in E and M, respectively. Two projections can be existed are $P: E \to N$ and $(I - P): E \to N^{\perp}$ where $Pz = u$ and $(I - P)z = v$. If $e_1, e_2, \dots, e_n$ are a basis of the space N, thus any element $z \in E$ can be written in the unique form:

$$z = u + v, \;\; u \in N, \;\; v \in N^{\perp}, \;\; u = \sum_{i=1}^{n} x_i e_i \,. \tag{11}$$

Likewise,, there are exists two projections are $Q: M \to \tilde{N}$ and $(I - Q\,): M \to \tilde{N}^{\perp}$ where

$$QG(z, \lambda) = G_1(z, \lambda) \quad \text{and} \quad (I - Q)G(z, \lambda) = G_2(z, \lambda)\,. \tag{12}$$

If $g_1, g_2, \dots, g_n$ are a basis of the space $\tilde{N}$, then

$$G(z, \lambda) = G_1(z, \lambda) + G_2(z, \lambda), \tag{13}$$

$$G_1(z, \lambda) \in \tilde{N}, \;\; G_2(z, \lambda) \in \tilde{N}^{\perp}\,, \tag{14}$$

$$G_1(z, \lambda) = \sum_{i=1}^{n} v_i(z, \lambda) g_i \,, \quad G_2(z, \lambda) \perp \tilde{N} \;. \tag{15}$$

Thus

$$G(z, \lambda) = QG(z, \lambda) + (I - Q)G(z, \lambda) = 0 \tag{16}$$

Hence

$$QG(z, \lambda) = 0 \tag{17}$$

$$(I - Q)G(z, \lambda) = 0 \tag{18}$$

or

$$QG(u + v, \lambda) = 0 \tag{19}$$

$$(I - Q)G(u + v, \lambda) = 0. \tag{20}$$

From implicit function theorem, there is a smooth map $\Theta: N \to N^{\perp}$ where, $\Theta(u,\lambda) = v$ and

$$(I - Q)G(u + \Theta(u,\lambda), \lambda) = 0. \quad (21)$$

To find the solutions of the equation $G(z,\lambda) = 0$ in the neighborhood of the point $z = a$, it is sufficient to find the solutions of the equation:

$$QG(u + \Theta(u,\lambda), \lambda) = 0 \quad (22)$$

Equation (22) is called bifurcation equation.

3. Results and discussion

3.1 Modify Lyapunov-Schmidt method (MLSM)

Modify Lyapunov-Schmidt method (MLSM) is similar to Lyapunov-Schmidt reduction (LSR) which is based on finding the homogeneous solution of the operator Eq.1 unlike MLSM. However, modify Lyapunov-Schmidt method is a procedure for obtaining the nonlinear Ritz approximation to a Fredholm functional which gives by:

Suppose that the nonlinear operator which is Fredholm with zero index $f: E \to F$ where

$$f(u,\gamma) = \Psi,\ \gamma \in R^n, u \in \Lambda \subset E \quad (23)$$

where E and F are real Banach space, $\Psi = \varepsilon\varphi$ is function (continuous function), ε is small parameter and $\Lambda \subseteq E$ is an open. Let's the operator f possess a variational property which means there is a functional $V: \Lambda \subset E \to R$ where $f = grad_H V$ when Λ is a bounded domain. The operator f is given by:

$$f(u,\gamma) = \mathrm{H}u + Nu = \Psi, \Psi \in \mathrm{F} \quad (24)$$

where $H = \frac{\partial f}{\partial u}(u_0,\gamma)$ represents the Frechet derivative of the operator f about u_0 and it's linear continuous Fredholm operator and N is the nonlinear operator for f. By application LSR, the decomposition is:

$$E = W \oplus W^{\perp},\ F = \widehat{W} \oplus \widehat{W}^{\perp} \quad (25)$$

When $W = ker\ H$ symbolizes to the null space of the operator f then $dim\ W = dim\widehat{W} = 2$. The orthogonal complements of the subspace W and $\widehat{W}$ are $W^{\perp}$ and $\widehat{W}^{\perp}$. If e_1 and e_2 are an orthonormal set in W where $He_i = \alpha_i(\gamma)e_i$, $\alpha_i(\gamma)$ is continuous function, where $i = 1,2$ then, $\forall\, u \epsilon E$ can be given in the unique below format:

$$u = w + v, \quad w = \xi_1 e_1 + \xi_1 e_2 \in W, \quad W \perp v \in W^{\perp}, \; \xi_i = \langle u, e_i \rangle, \tag{26}$$

When $\langle .,. \rangle$ represents the inner product thus, the projections $p: E \to W$ & $I - p: E \to W^{\perp}$ are defined by:

$\omega = pu$ & $(I - p)u = v$. Similarly, the exist two projections of $Q: F \to W$ and $I - Q: F \to \widehat{W}^{\perp}$ are defined by:

$$f(u,\gamma) = Qf(u,\gamma) + (I - Q)f(u,\gamma)$$

Or (27)

$$f(\omega + v,\gamma) = Qf(\omega + v,\gamma) + (I - Q)f(\omega + v,\gamma)$$

Thus, can be got:

$$Qf(\omega + v,\gamma) = \Psi_1, \quad \Psi_1 \in W \tag{28}$$

$$(I - Q)f(\omega + v,\gamma) = \Psi_2, \; \Psi_2 \in \widehat{W}^{\perp} \tag{29}$$

Where $\Psi = \Psi_1 + \Psi_2$, $\Psi_1 = t_1 e_1 + t_2 e_2$ with assuming that,

$$\Psi_2 = a_1 {t_1}^2 + a_2 t_1 t_2 + a_3 {t_2}^2 \tag{30}$$

By implicit function theorem getting

$$M(\xi,\beta) = V(\theta(\xi,\beta),\beta), \; \xi = (\xi_1, \xi_2, \dots, \xi_n) \tag{31}$$

Where $deg\, M \geq 2$ and the functional V has the linear Ritz approximation represented by the function M that defined by:

$$M(\xi,\beta) = V(\textstyle\sum_{i=1}^{n} \xi_i e_i, \beta) = M_0(\xi) + M_1(\xi,\beta) \tag{32}$$

Where $M_0(\xi)$ is a homogenous polynomial with degree of $n \geq 3$ s.t $M_0(0) = 0$ & $M_1(\xi,\beta)$ is a polynomial function of degree $< n$. If $q1, q2, \dots\, qm$ are the coefficients to the square terms of the function $M_1(\xi,\beta)$ thus the function $M_1(\xi,\beta)$ can be written in the formula:

$$M_1(\xi,\beta) = M_2(\xi,\beta) + \textstyle\sum_{k=1}^{m} q_k \xi_k^2 \tag{33}$$

Where $deg\, M_2 = d, 2 < d < n.$

The functional V has nonlinear Ritz approximation and the function M is defined by:

$$M(\xi,\beta) = V(\sum_{i=1}^{n} \xi_i e_i + \theta(\sum_{i=1}^{n} \xi_i e_i, \beta), \beta) \quad (34)$$

When $\theta(\omega,\beta) = v(x,\xi,\beta), v \in N^{\perp}$. Taylor's expansion to $\mu_k(\xi)$ and $v(x,\xi,\beta)$ can be employed to get the nonlinear Ritz approximation for the functional V by assuming::

$$q_k = \hat{q}_k + \mu_k(\xi) = \hat{q}_k + \sum_{i=2}^{r} D_k^j(\xi), \quad k = 1, \dots, m \quad (35)$$

$$v(x,\xi,\beta) = \sum_{i=2}^{r} B^j(\xi) \quad (36)$$

Where $D_k^{(j)}(\xi)$ and $B^{(j)}(\xi)$ are homogenous polynomials with degree j , have coefficients are μ_{ki} and $v_{ji}(x,\beta)$, respectively and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$.

$$Qf(u,\gamma) = \langle f(u,\gamma), e_1\rangle e_1 + \langle f(u,\gamma), e_2\rangle e_2 = \Psi_1 \quad (37)$$

And

$$\langle Hu + Nu, e_1\rangle e_1 + \langle Hu + Nu, e_2\rangle e_2 = \Psi_1 \quad (38)$$

Hence

$$q_1\xi_1 e_1 + q_2\xi_2 e_2 + \langle Nu, e_1\rangle e_1 + \langle Nu, e_2\rangle e_2 = \Psi_1, \qquad q_i = \alpha_i(\gamma) \quad (39)$$

Or

$$q_1\xi_1 e_1 + q_2\xi_2 e_2 + [\int_\Omega N(w+v)e_1]e_1 + [\int_\Omega N(w+v)e_2]e_2 = \Psi_1, \quad (40)$$

From equation 27 can be obtained:

$$(I-Q)f(u,\gamma) = f(u,\gamma) - Qf(u,\gamma) \quad (41)$$

From $H(\omega+v) + N(\omega+v) = \Psi_2$ it follows that

$$Hv + N(w,v) + q_1\xi_1 e_1 + q_2\xi_2 e_2 = \Psi_2, \quad (42)$$

Substituting the values of $q_i, \mu_i(\xi)$ and $v(x,\xi,\delta)$ in equation 40&42 yields;

$$\left[\hat{q}_1 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_1 e_1 + \left[\hat{q}_2 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_2 e_2 + \left[\int_\Omega N(q_1\xi_1 e_1 + q_2\xi_2 e_2 + \sum_{j=2}^{r} B^j(\xi)e_1\right] e_1 + \left[\int_\Omega N(q_1\xi_1 e_1 + q_2\xi_2 e_2 \ \sum_{j=2}^{r} B^j(\xi)e_2\right] e_2 = \Psi_1 \quad (43)$$

$$\mathrm{H}(\sum_{j=2}^{r} B^{j}(\xi)) + N(q_1\xi_1 e_1 + q_2\xi_2 e_2 + \sum_{j=2}^{r} B^{j}(\xi)) + [\hat{q}_1 + \sum_{j=2}^{r}(D_1^{j}(\xi) + D_2^{j}(\xi))]\xi_1 e_1 + [\hat{q}_2 + \sum_{j=2}^{r}(D_1^{j}(\xi) + D_2^{j}(\xi))]\xi_2 e_2 = \Psi_2 \quad (44)$$

To calculate the functions $v(\mathrm{x}, \xi, \beta)$ & $\mu_k(\xi)$, let the coefficients of $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ in Eq.43 to find the value of μ_{ki}. From Eq.44 can be got a linear ODE in the variable $v_{ji}(x, \gamma)$. Solving the equation led to get the value of $v_{ji}(x, \gamma)$.

3.2 Applications

The Modify Lyapunov-Schmidt method (MLSM) that given in the previous section to get nonlinear Ritz approximation for the functional corresponding to the nonhomogeneous wave equation that describes the motion and oscillations of the elastic beam which can be described by the partial nonlinear differential equation:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta u + u^3 = 0 \quad (45)$$

The $u(x, t)$ is describe a dimensionless deviation of the beam thus, the ingredients $(x, t) \in [0,1] \times (0, \infty)$ are represented the dimensionless of space and time variables while α and β are parameters depict the beam's tension and constant in charge for restoring the features of the support, respectively. However, Eq 45 can be converted to the order differential equation with variable $w(x)$ by chose $(w(x) = u(x, t))$ as follows:

$$\frac{d^4 w}{dx^4} + \alpha\frac{d^2 w}{dx^2} + \beta w + w^3 = \psi \quad (46)$$

Equation 46 can be studied with the boundary condition as below:

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0 \quad (47)$$

The nonlinear Ritz approximation can be obtained through below theory:

Theorem 3.2.1. The functional

$$\mathrm{V}(\mathrm{w}, \lambda, \psi) = \int_0^{\pi}(\frac{(\mathrm{w}'')^2}{2} - \alpha\frac{(\mathrm{w}')^2}{2} + \beta\frac{\mathrm{w}^2}{2} + \frac{\mathrm{w}^4}{4} - \mathrm{w}\psi)dx. \quad (48)$$

has the key function of the form

$$\widehat{W}(\xi,\delta) = \xi_1{}^{12} + \xi_2{}^{12} + \lambda_1\xi_1{}^2\xi_2{}^{10} + \lambda_2\xi_1{}^4\xi_2{}^8 + \lambda_3\xi_1{}^6\xi_2{}^6 + \lambda_4\xi_1{}^8\xi_2{}^4 + \lambda_5\xi_1{}^{10}\xi_2{}^2 + \lambda_6\xi_1{}^2\xi_2{}^8 + \lambda_7\xi_1{}^8\xi_2{}^2 + \lambda_8\xi_1{}^6\xi_2{}^4 + \lambda_9\xi_1{}^4\xi_2{}^6 + \lambda_{10}\xi_1{}^8 + \lambda_{11}\xi_2{}^{11} + \lambda_{12}\xi_1{}^6\xi_2{}^2 + \lambda_{13}\xi_1{}^2\xi_2{}^6 + \lambda_{14}\xi_1{}^4\xi_2{}^4 + \lambda_{15}\xi_1{}^6 + \lambda_{16}\xi_2{}^6 + \lambda_{17}\xi_1{}^4\xi_2{}^2 + \lambda_{18}\xi_1{}^2\xi_2{}^4 + \lambda_{19}\xi_1{}^4 + \lambda_{20}\xi_2{}^4 + \lambda_{21}\xi_1{}^2\xi_2{}^2 + \gamma_1\xi_1{}^3 + \gamma_2\xi_1\xi_2{}^2 + \lambda_{22}\xi_1{}^2 + \lambda_{23}\xi_2{}^2 + t_1\xi_1 + t_2\xi_2 + O(|\zeta|^{12}) + O(|\zeta|^{12})O(|\delta|) \quad (49)$$

$\lambda_i = \lambda_i(\alpha,\beta), i = 1,2,\dots.23, \gamma_i = \gamma_i(\alpha,\beta,t), i = 1,2$ and $\xi = (\xi_1,\xi_2), \delta = (\gamma_i,\lambda_i)$ such that λ,γ are parameters.

Proof. Obtaining the nonlinear approximation requires writing Eq. 46 as a nonlinear Fredholm operator as:

$$f(w,\lambda) = \frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + \beta w + w^3, \quad (50)$$

Where $E = C^4([0,\pi],R)$ is the space of each continuous functions that have differential of order mostly four, $F = \{f \mid f:[0,\pi] \to R \text{ is continuous function}\}$, and $w = w(x), x \in [0,\pi], \lambda = (\alpha,\beta)$. Every solution of Eq.50 is a solution of operator equation.

$$f(w,\lambda) = \psi, \psi \in F \quad (51)$$

Note that the operator f owns variational property

$$V(w,\lambda,\psi) = \int_0^{\pi}\left(\frac{(w'')^2}{2} - \alpha\frac{(w')^2}{2} + \beta\frac{w^2}{2} + \frac{w^4}{4} - w\psi\right)dx \quad (52)$$

In this case, the critical points of the functional in Eq.52 are the solutions of Eq. 50.

Thus, the study of boundary value of problem 46 is equivalent to study the extremal problem

$$V(w,\lambda,\psi) \to extr, w \in E, \lambda \in R^n \quad (53)$$

Through using Lyapunov-Schmidt method to reduce into finite dimensional space, analysis of bifurcation can be found. The localized parameters, $\alpha = \alpha_1 + \delta_1, \beta = \beta_1 + \delta_2, \delta_1, \delta_2$ are small parameters.

The reduction is led to get the function in two variables are:

$$W(\xi,\delta) = \inf\ V(w,\delta), \quad (54)$$

$$\xi = (\xi_1, \xi_2), \delta = (\delta_1, \delta_2) \tag{55}$$

Is well known that the reduction of Lyapunov—Schmidt function $W(\xi,\delta)$ is smooth. This function has all the topological and analytical properties of functional V [1]. In particular, for small δ there is one-to-one corresponding between the critical points of functional V and smooth function W that preserve the type of critical points (multiplicity, index Morse, etc…) [1]. Functional V is even, $V(-w,\lambda,0) = V(w,\lambda\ \ 0)$ and symmetric. By using (LSR) is led to obtain the linearized equation corresponding to Eq. 46 that given by:

$$h'''' + \alpha h'' + \beta h = 0\,, h \in E, \tag{56}$$

$$h(0) = h(\pi) = h''(0) = h''(\pi) = 0 \tag{57}$$

The last equation gives in the characteristic lines ($\alpha\beta - plane$) thus, a point of characteristic lines it's the points of (α,β) where the Eq. 50 has nontrivial solutions. The bifurcation point [1] can be found in the space of parameters(α,β) from the point of intersection $\alpha\beta - plane$. Therefore, $(\alpha,\beta) = (5,4)$ is a bifurcation point for the boundary value problem 50. However, the result of this intersection led to obtain bifurcation along the modes $e_1 = c_1\, sin\,(x)$, $e_2 = c_2 \sin(2x)$. Localization parameters are given by:

$$\breve{\alpha} = 5 + \delta_1, \breve{\beta} = 4 + \delta_2. \tag{58}$$

Lead to the bifurcation along the modes, e_1, e_2 where $\|e_1\| = \|e_2\| = 1$ and $c_1 = c_2 = \sqrt{\frac{2}{\pi}}$.

suppose that $N = \ker(A) = span\{e_1, e_2\}$, where $A = f_w(0,\lambda) = \frac{d^4}{dx^4} + \alpha\frac{d^2}{dx^2} + \beta$ then, the space E can be separate into two subspaces are N and orthogonal complement to N.

$$E = N \oplus \hat{E}, \hat{E} = N^{\perp} \cap E = \{v \in E: v \perp N\}, \tag{59}$$

Likewise, the space F can be separated into two subspaces are M, and orthogonal complement to M as follows:

$$F = N \oplus \hat{F},\ \hat{F} = M^{\perp} \cap F = \{v \in F: v \perp M\} \tag{60}$$

$P: E \to N$ and $I - P: E \to \hat{E}$ are projections where $pw = u$ and $(I-P)w = v$ thus, each element $w \in E$ can be represented by $w = u + v, u = \sum_{i=1}^{2} \xi_i e_i\,, N \perp v \in \hat{E}, \xi_i = \langle w, e_i\rangle$ similarly there exists projection $Q: F \to N$ and $I - Q: F \to \hat{F}$ in which

$$f(u,\lambda) = Qf(u,\lambda) + (I-Q)f(u,\lambda) = \psi \quad (61)$$

Accordingly, Eq. 23.27 can be represented by:

$$Qf(u+v,\lambda) = \psi_1 \quad (62)$$

$$(I-Q)f(u+v,\lambda) = \psi_2 \quad (63)$$

Where $\psi_1 = e_1 t_1 + e_2 t_2$ *and* $\psi_2 = a_1 {t_1}^2 + a_2 t_1 t_2 + a_3 {t_2}^2$

By IFT a smooth map $\phi: N \to \hat{E}$ can be found.

$$W(\xi,\delta,\psi) = V(\phi(\xi,\lambda),\delta,\psi), \delta = (\delta_1,\delta_2) \quad (64)$$

The functional V has a nonlinear Ritz approximation is given by following function:

$$W(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2 + \phi(\xi_1 e_1 + \xi_2 e_2,\delta),\delta) \quad (65)$$

$$\phi(u,\delta) = v(x,\delta,\lambda) \quad (66)$$

Determine the function $W(\xi,\delta)$ of the functional V require find the functions $v(x,\xi,\lambda) = O(\xi^3)$, $\mu(\xi^2) = O(\xi), \tilde{\mu}(\xi) = O(\xi^2), \xi = (\xi_1,\xi_2)$ as a power series in term of ξ, where

$$q_1 = \widetilde{q_1} + \mu(\xi_1,\xi_2), q_2 = \widetilde{q_2} + \tilde{\mu}(\xi_1,\xi_2)$$

$$\left.\begin{aligned} v(x,\xi,\lambda) = v_0(x,\lambda)\xi_1^3 + v_1(x,\lambda){\xi_1}^2\xi_2 + v_2(x,\lambda)\xi_1{\xi_2}^2 + v_3(x,\lambda)\xi_2^3 + \cdots \\ \mu(\xi_1,\xi_2) = \mu_0{\xi_1}^2 + \mu_1\xi_1\xi_2 + \mu_2{\xi_2}^2 \\ \tilde{\mu}(\xi_1,\xi_2) = \tilde{\mu}_0{\xi_1}^2 + +\tilde{\mu}_1\xi_1\xi_2 + \tilde{\mu}_2{\xi_2}^2 \end{aligned}\right\} \quad (67)$$

Equation 18 can be splitted as follows

$$f(u,\lambda) = Au + Tu = \psi, \quad (68)$$

where $Aw = \frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + \beta w$ is a linear part while $Tw = w^3$ is a nonlinear part of Eq.51. Since,

$$Qf(w,\lambda) = \sum_{i=1}^{2}\langle f(w,\lambda), e_i\rangle e_i = \psi_1, \quad (69)$$

we have

$$\sum_{i=1}^{2}\langle A(w) + T(w), e_i\rangle e_i = \sum_{i=1}^{2}\left(\int_0^{\pi}(A(w)e_i + T(w)e_i)dx\right)e_i = \psi_1. \quad (70)$$

Thus

$$\left(q_1\xi_1+\int_0^{\pi}(\xi_1e_1+\xi_2e_2+v)^3e_1dx\right)e_1+\left(q_2\xi_2+\int_0^{1}(\xi_1e_1+\xi_2e_2+v)^3e_2dx\right)e_2=e_1t_1+e_2t_2 \quad (71)$$

and

$$v''''+\alpha v''+\beta v+(\xi_1e_1+\xi_2e_2+v)^3+q_1\xi_1e_1+q_2\xi_2e_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2 \quad (72)$$

Substituting $q_1=\widetilde{q_1}+\mu(\xi_1,\xi_2)$ and $q_2=\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)$ in Eqs. 71 &72, led to get

$$\left[\left(\widetilde{q_1}+\mu(\xi_1,\xi_2)\right)\xi_1+\xi_1^3\int_0^{\pi}e_1^4dx+3{\xi_1}^2\xi_2\int_0^{\pi}e_1^3e_2dx+3\xi_1{\xi_2}^2\int_0^{\pi}e_1^2{e_2}^2dx+\xi_2^3\int_0^{\pi}e_1e_2^3dx\right]e_1+\left[\left(\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\right)\xi_2+\xi_1^3\int_0^{\pi}e_1^3e_2dx+3{\xi_1}^2\xi_2\int_0^{\pi}e_1^2{e_2}^2dx+3\xi_1{\xi_2}^2\int_0^{\pi}e_1e_2^3dx+\xi_2^3\int_0^{1}e_2^4dx\right]e_2=e_1t_1+e_2t_2 \quad (73)$$

$$v''''+\alpha v''+\beta v+\xi_1^3e_1^3+3{e_1}^2e_2{\xi_1}^2\xi_2+3e_1{e_2}^2\xi_1{\xi_2}^2+\xi_2^3e_2^3+v^3+3v^2e_1\xi_1+3v^2e_2\xi_2+3v{e_1}^2{\xi_1}^2+6ve_1e_2\xi_1\xi_2+3v{e_2}^2{\xi_2}^2+\left(\widetilde{q_1}+\mu(\xi_1,\xi_2)\right)\xi_1e_1+\left(\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\right)\xi_2e_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2 \quad (74)$$

The functions $v(x,\xi,\lambda)$, $\mu(\xi)$ and $\tilde{\mu}(\xi)$ in Eq.67 are determine by finding the coefficients $\mu_0,\mu_1,\mu_2,$ and $\tilde{\mu}_0,\tilde{\mu}_1,\tilde{\mu}_2,v_0,v_1,v_2,and\ v_3$ in Eqs. 73 & 74.Equal the coefficients of ξ_1^3 in the Eqs. 73&74 led to obtain two equations,

$$\left[\mu_0+\int_0^{\pi}e_1^4dx\right]e_1+\left[\int_0^{\pi}e_1^3e_2dx\right]e_2=0 \quad (75)$$

$$v_0''''+\alpha v_0''+\beta v_0+e_1^3+\mu_0e_1=0. \quad (76)$$

The Eq.75 gives $\mu_0=-\frac{3}{2\pi}$ that substitute in ODE 76.

$$v_0''''+\alpha v_0''+\beta v_0+e_1^3-\frac{3}{2\pi}e_1=0 \quad (77)$$

And

$$v_0=\frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(81-9\alpha+\beta)}sin(3x) \quad (78)$$

Now, from the coefficients of ${\xi_1}^2\xi_2$ can be got

$$v_1=\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(256-16\alpha+\beta)}sin(4x) \quad (79)$$

Equating the coefficients of $\xi_1\xi_2^2$ is led to

$$v_2 = -\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(81-9\alpha+\beta)}sin(3x) + \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(625-25\alpha+\beta)}sin(5x) \tag{80}$$

Equating the coefficients of ${\xi_2}^3$ is led to obtain

$$v_3 = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(1296-36\alpha+\beta)}sin(6x) \tag{81}$$

Now, the nonlinear approximation solutions of Eq.51 are determined by substituting the values of $\mu_0, \mu_1, \mu_2, \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, v_0, v_1, v_2\ and\ v_3$ in Eq.67.

$$w(x,\xi) = \sqrt{\frac{2}{\pi}}\xi_1 sin(x) + \sqrt{\frac{2}{\pi}}\xi_2 sin(2x) + \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{\xi_1^3}{(81-9\alpha+\beta)}sin(3x)$$

$$+\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(256-16\alpha+\beta)}sin(4x)\,{\xi_1}^2\xi_2 + \left[\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(81-9\alpha+\beta)}sin(3x)\ + \right.$$

$$\left.\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(625-25\alpha+\beta)}sin(5x)\right]\xi_1\xi_2^2 + \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(1296-36\alpha+\beta)}sin(6x) + O(\xi^5), \tag{82}$$

$$q_1 = \widetilde{q_1} - \frac{3}{2\pi}{\xi_1}^2 + \frac{3}{\pi}{\xi_2}^2 + O(\xi^3), \tag{83}$$

$$q_2 = \widetilde{q_2} - \frac{3}{\pi}{\xi_1}^2 + \frac{3}{2\pi}{\xi_2}^2 + O(\xi^3), \tag{84}$$

However, substituting Eq.82 in the functional $V(u,\lambda)$ led to get the function $\widehat{W}(\xi,\delta)$.

Moreover, the functional V has a linear Ritz approximation that given by the function W:

$$W(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2, \delta) = \xi_1^4 + 4{\xi_1}^2{\xi_2}^2 + {\xi_2}^4 + \frac{q_1}{2}{\xi_1}^2 + \frac{q_2}{2}{\xi_2}^2 t_1\xi_1 + t_2\xi_2 \tag{85}$$

4. Conclusions

This paper presented the modify Lyapunov-Schmidt reduction for nonhomogeneous problems when the dimension of the null space is equal to two. the modify Lyapunov-Schmidt reduction was applied to find a nonlinear Ritz approximation for Fredholm functional defined by the nonhomogeneous nonlinear differential equations like elastic beams equation. The results in this study show that MLSR is a fast and successful technique to find nonlinear Ritz approximation of partial differential equations. Therefore, MLSR is an effective and accurate method for solving the nonlinear problems that appear in mathematics, Physics and Engineering.

References

[1] Y.I. Sapronov, S..L Tsarev, Global comparison of finite-dimensional reduction schemes in smooth variational problems, Math Notes, 67(2000) 631–638. https://doi.org/10.1007/BF02676336

[2] M. M. Vainberg, V. A. Trenogin, Theory of the branching of solutions of nonlinear equations, Leyden, Noordhoff International Pub, 24(1974) 478-485. https://doi.org/10.1137/1018033

[3] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Equations, J. Appl. Math. Mech., 45 (1964) 380-395. https://doi.org/10.1002/zamm.19640441041

[4] S.N. Chow, J.K. Hale, Methods of Bifurcation Theory, N. Y. Springer- Verlag, (1982), https://doi.org/10.1007/978-1-4613-8159-4

[5] Y.I. Sapronov, B.M. Darinskii, S.L. Tcarev, Bifurcations of Extremals of Fredholm Functionals, J. Math. Sci., 145 (2007) 5311-5453, https://doi.org/10.1007/s10958-007-0356-2.

[6] A.A. Mizeal, M.A. Abdul Hussain, Two-Mode Bifurcation in Solution of a Perturbed Nonlinear Fourth Order Differential Equation, Arch. Math., 48 (2012) 27-37. https://doi.org/10.5817/AM2012-1-27

[7] M. A. Abdul Hussain, Two modes bifurcation solutions of elastic beams equation with nonlinear approximation. Commun. Math. Appl., 1 (2010) 123-131. https://doi.org/10.26713/cma.v1i2.119

[8] M.J. Mohammed, Lyapunov-Schmidt Method in Bifurcation Solutions of Nonlinear Fourth Order Differential Equation, J. Thi-Qar Sci, 3 (2011) 135-145. https://doi.org/10.32792/utq/utjsci/vol3/1/15

[9] Z. A. Shawi, M.A. Abdul Hussain, Bifurcation Solutions of Fourth Order Non-linear Differential Equation Using a Local Method of Lyapunov –Schmidt, Bas. J. Sci., 38 (2020) 221–233. https://doi.org/10.29072/basjs.202124

[10] H.K. Kadhim, M.A. Abdul Hussain, The analysis of bifurcation solutions of the Camassa–Holm equation by angular singularities. Probl Anal Issues Anal., 9 (2020) 66–82, https://doi.org/10.15393/j3.art.2020.6770

[11] Z.S. Madhi, M.A. Abdul Hussain, Bifurcation Diagram of $W(u_j, \tau)$-function with (p, q) −parameters, Iraqi J. Sci., 63 (2022) 667-674, https://doi.org/10.24996/ijs.2022.63.2.23

طريقة لبينوف- شمدت المعدلة للمعادلات التفاضلية غير المتجانسة

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المستخلص

في هذا البحث قدمنا طريقة ليابونوف- شمدت المعدلة في حالة المسائل غير المتجانسة عندما يكون بعد الفضاء الصفري مساو الى اثنان .تم اختبار الطريقة الجديدة من خلال استخدامها للحصول على حلول التقريب لمعادلة الموجة غير الخطية .تم العثور على الدالة الرئيسية المقابلة للدالة المتعلقة بهذه المعادلة.