Investigation of Fractional Spline Function-Based Lacunary Interpolation with Convergence Analysis

Ridha Ghafoor Kareem[1,*], Karwan H. F. Jwamer[2], Faraidun K. Hamasalh[3]

1. Department of Mathematical Science, College of basic Education, University of Sulaimani, Kurdistan Region, IRAQ
2. Department of Mathematics, College of Science, University of Sulaimani, Kurdistan Region, IRAQ
3. Bakrajo Technical Institute, Sulaimani Polytechnic University Kurdistan Region, IRAQ

*Corresponding author E-mail: ridha.karem@univsul.edu.iq

https://doi.org/10.29072/basjs.20240213

ARTICLE INFO

Keywords

Spline Function; Fractial Function; Error Bounds; Converge Analysis

ABSTRACT

This paper presents a derived fractional degree (1/2, 3/2, 5/2) lacunary interpolation technique. We demonstrate the proper representation of complex functions and their evolution throughout the specified interval utilizing the three-spline function. The advanced modified extended spline technique is employed to achieve many types of boundary conditions, including fractional order brightness. The impact of the fractional derivative on the model is illustrated through convergence analysis simulated for different values of beta. Issues related to spline functions can be addressed using this interpolation method for the construction of spline functions.

Received 14 Nov 2024; Received in revised form 13 Dec 2024; Accepted 16 Dec 2024, Published 31 Dec 2024

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

1. Introduction

Lacunary interpolation using fractional degree spline functions is a specialized area within numerical analysis, focusing on approximating functions through splines of fractional order. This approach is particularly useful for solving fractional differential equations, which arise in various fields such as physics, engineering, and finance. Lacunary interpolation involves constructing an interpolating function that is defined only at a subset of points, known as lacunae. This method is advantageous when data is sparse or when certain values are missing. The use of spline functions, which are piecewise polynomial functions, allows for smooth approximations between these points. Fractional degree splines extend traditional spline functions by allowing for non-integer degrees, which can provide greater flexibility and accuracy in approximation. The fractional degree splines are particularly effective in capturing the behavior of functions that exhibit fractional dynamics, which are common in real-world applications. The existence of a unique spline function that satisfies certain conditions is crucial. Theorems have been established to demonstrate the conditions under which these splines exist and are unique, particularly for fractional orders. An essential aspect of lacunary interpolation is the analysis of error bounds. This involves establishing how closely the interpolating spline approximates the actual function, which can be quantified through various mathematical theorems. The methods of lacunary interpolation

using fractional degree splines have been applied to solve fractional differential equations numerically. The results show that these methods can yield accurate approximations, making them suitable for various scientific and engineering applications. Recent research has focused on extending the capabilities of lacunary interpolation with higher-degree splines. Studies have shown that using higher-degree splines can improve the accuracy of the interpolation significantly, especially in complex applications involving fractional calculus. Refer to the survey article [1-4] for the full background information. To solve differential equations of fractional Order, Faraidun K. Hamasalh [5-6] found the error bounds to fractional polynomial spline in a work. We used the identical lacunry data in the current work, and we demonstrated that the error bounded more accurately than Karwan H. F. Jwamer [7]. The similar method was applied by several writers, albeit with different lacunary data; for instance, see [8-11]. The structure of this work is as follows: The degree three spline function that interpolates the lacunary data $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$ is defined. A few theoretical

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

findings about the uniqueness and existence of the degree three spline functions are presented, along with a study of convergence analysis [12-16].

We present a three degree spline interpolation for one dimensional and given sufficiently smooth function $f(x)$ defined on I=[0,1] denote the uniform partition of I with $\delta: 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a consistent division of the interval [0,1], when $x_i = ih$, $i = 0,1, \dots, n$;

$nh = 1$. The class of spline functions $S_p(3,3,n)$ is defined as follows:

{ $S_p(3,3,n)$ is the class of spline functions $S_\delta(x)$ such that $S_\delta(x)$ is a polynomial of degree less than or equal to 3 on each sub interval $[x_i, x_{i+1}]$, $i = 0,1, \dots, n-1$ and $S_\delta(x) \in C^3[0,1]$ with n knots}. The class of spline functions Sp(3,3, n) any element $S_\delta(x) \in S_p(3,3,n)$in this work satisfies the following requirements:

$$s_i^{(p)}(x_i) = f^{(p)}(x_i) = y_i^{(p)}. p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ; i = 0,1, \dots, n-1, \quad \text{(i)}$$

$$s_{n-1}^{(p)}(x_n) = f^{(p)}(x_n) = y_n^{(p)}. p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \quad \text{(ii)}$$

$$s_i(x_{i+1}) = s_{i+1}(x_{i+1}) = f(x_{i+1}) = y_{i+1}; i = 0,1, \dots, n-2, \quad \text{(iii)}$$

$$s_i^{(p)}(x_{i+1}) = s_{i+1}^{(p)}(x_{i+1}) = f^{(p)}(x_{i+1}) = y_{i+1}^{(p)} . p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ; i = 0,1, \dots, n-2, \quad \text{(iv)}$$

$$s_0(x_0) = f(x_0) = y_0 , s_{n-1}(x_n) = f(x_n) = y_n . \quad \text{(v)}$$

2. Existence and uniqueness

This section covers the existence and originality of our approach, which we now discuss. The similar error estimate for the much less smooth class of functions $\in C^3[a, b]$, can be computed as follows in terms of the modulus of continuity under uniform partition and simplified limited condition:

Theorem 1: There is only a unique spline function $S_\delta(x) \in S_q(3,3,n)$ that satisfies conditions i through v.

Proof: If it's feasible, let's say that

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$S_\delta(x)=\begin{cases} S_0(x) \text{ when } x_0\le x\le x_1 , \\ S_i(x) \text{ when } x_i\le x\le x_{i+1}, \quad i=1,\dots,n-2, \\ S_{n-1}(x) \text{ when } x_{n-1}\le x\le x_n .\end{cases}$$

subsequently as a result of the situations (i–v). We possess the ability to write.

$$s_0(x)=y_0+\frac{(x-x_0)^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}}y_0^{\left(\frac{1}{2}\right)}+(x-x_0)b_{0,1}+\frac{(x-x_0)^{\frac{3}{2}}}{\frac{3\sqrt{\pi}}{4}}y_0^{\left(\frac{3}{2}\right)}+(x-x_0)^2b_{0,2}$$

$$+\frac{(x-x_0)^{\frac{5}{2}}}{\frac{15\sqrt{\pi}}{8}}y_0^{\left(\frac{5}{2}\right)}+(x-x_0)^3\,b_{0,3}\,, \quad \text{(vi)}$$

$$s_i(x)=b_{i,0}+\frac{(x-x_i)^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}}y_i^{\left(\frac{1}{2}\right)}+(x-x_i)b_{i,1}+\frac{(x-x_i)^{\frac{3}{2}}}{\frac{3\sqrt{\pi}}{4}}y_i^{\left(\frac{3}{2}\right)}+(x-x_i)^2b_{i,2}$$

$$+\frac{(x-x_i)^{\frac{5}{2}}}{\frac{15\sqrt{\pi}}{8}}y_i^{\left(\frac{5}{2}\right)}+(x-x_i)^3\,b_{i,3}\,, \quad \text{(vii)}$$

$$s_{n-1}(x)=b_{n-1,0}+\frac{(x-x_{n-1})^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}}y_{n-1}^{\left(\frac{1}{2}\right)}+(x-x_{n-1})b_{n-1,1}+\frac{(x-x_{n-1})^{\frac{3}{2}}}{\frac{3\sqrt{\pi}}{4}}y_{n-1}^{\left(\frac{3}{2}\right)}$$

$$+(x-x_{n-1})^2b_{n-1,2}+\frac{(x-x_{n-1})^{\frac{5}{2}}}{\frac{15\sqrt{\pi}}{8}}y_{n-1}^{\left(\frac{5}{2}\right)}+(x-x_{n-1})^3\,b_{n-1,3}\,, \quad \text{(viii)}$$

Using (iii) and (iv) with $p=i=0$, obtain the coefficients in $s_0(x)$. We get the following linear system of equations:

$$b_{0,1}+\frac{4}{3}h\,b_{0,2}+\frac{8}{5}h^2\,b_{0,3}=\frac{\sqrt{\pi}}{2}h^{\frac{-1}{2}}[y_1^{\left(\frac{1}{2}\right)}-y_0^{\left(\frac{1}{2}\right)}-hy_0^{\left(\frac{3}{2}\right)}-\frac{h^2}{2}y_0^{\left(\frac{5}{2}\right)}]\,, \quad \text{(ix)}$$

$$b_{0,2}+2\,h\,b_{0,3}=\frac{\sqrt{\pi}}{4}h^{-\frac{1}{2}}[y_1^{\left(\frac{3}{2}\right)}-y_0^{\left(\frac{3}{2}\right)}-h\,y_0^{\left(\frac{5}{2}\right)}]\,, \quad \text{(x)}$$

$$b_{0,3}=\frac{\sqrt{\pi}}{12}h^{\frac{-1}{2}}(y_1^{\left(\frac{5}{2}\right)}-y_0^{\left(\frac{5}{2}\right)})\,, \quad \text{(xi)}$$

Solving these equations yields:

$$b_{0,1}=\frac{\sqrt{\pi}}{6}h^{\frac{-1}{2}}\left[3\left(y_1^{\left(\frac{1}{2}\right)}-y_0^{\left(\frac{1}{2}\right)}\right)-h\left(2y_1^{\left(\frac{3}{2}\right)}+y_0^{\left(\frac{3}{2}\right)}\right)+\frac{h^2}{30}\left(16y_1^{\left(\frac{5}{2}\right)}-y_0^{\left(\frac{5}{2}\right)}\right)\right], \quad \text{(xii)}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$b_{0,2} = \frac{\sqrt{\pi}}{4} h^{-\frac{1}{2}} \left[(y_1^{\left(\frac{3}{2}\right)} - y_0^{\left(\frac{3}{2}\right)}) - \frac{1}{3} h (2y_1^{\left(\frac{5}{2}\right)} + y_0^{\left(\frac{5}{2}\right)}) \right], \quad \text{(xiii)}$$

$$b_{0,3} = \frac{\sqrt{\pi}}{12} h^{\frac{-1}{2}} \left[y_1^{\left(\frac{5}{2}\right)} - y_0^{\left(\frac{5}{2}\right)} \right], \quad \text{(xiv)}$$

The coefficients in $s_i(x)$, $k = 0,1,\dots,n-2$ can be obtained using (i) we arrive at the following linear equation system.

$$b_{i+1,0} - b_{i,0} - hb_{i,1} - h^2 b_{i,2} - h^3\, b_{i,3} = \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{1}{2}\right)} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_i^{\left(\frac{3}{2}\right)} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_i^{\left(\frac{5}{2}\right)} \quad \text{(xv)}$$

$$b_{i,1} + \frac{4}{3} h\, b_{i,2} + \frac{8}{5} h^2\, b_{i,3} = \frac{\sqrt{\pi}}{2} h^{-\frac{1}{2}} [y_{i+1}^{\left(\frac{1}{2}\right)} - y_i^{\left(\frac{1}{2}\right)} - h y_i^{\left(\frac{3}{2}\right)} - \frac{h^2}{2} y_i^{\left(\frac{5}{2}\right)}], \quad \text{(xvi)}$$

$$b_{i,2} + 2h\, b_{i,3} = \frac{\sqrt{\pi}}{4} h^{-\frac{1}{2}} [y_{i+1}^{\left(\frac{3}{2}\right)} - y_i^{\left(\frac{3}{2}\right)} - h\, y_i^{\left(\frac{5}{2}\right)}] \quad , \quad \text{(xvii)}$$

$$b_{i,3} = \frac{\sqrt{\pi}}{12} h^{-\frac{1}{2}} (y_{i+1}^{\left(\frac{5}{2}\right)} - y_i^{\left(\frac{5}{2}\right)}) \quad . \quad \text{(xviii)}$$

Solving four equations yields:

$$b_{i,2} = \frac{\sqrt{\pi}}{4} h^{-\frac{1}{2}} (y_{i+1}^{\left(\frac{3}{2}\right)} - y_i^{\left(\frac{3}{2}\right)}) - \frac{\sqrt{\pi}}{12} h^{\frac{1}{2}} (2y_{i+1}^{\left(\frac{5}{2}\right)} + y_i^{\left(\frac{5}{2}\right)}), \quad \text{(xix)}$$

$$b_{i,1} = \frac{\sqrt{\pi}}{2} h^{-\frac{1}{2}} [(y_{i+1}^{\left(\frac{1}{2}\right)} - y_i^{\left(\frac{1}{2}\right)}) + \frac{1}{90} h^2 \left(16 y_{k+1}^{\left(\frac{5}{2}\right)} - y_i^{\left(\frac{5}{2}\right)} \right) - \frac{1}{3} h \left(2 y_{i+1}^{\left(\frac{3}{2}\right)} + y_i^{\left(\frac{3}{2}\right)} \right)], \quad \text{(xx)}$$

$$b_{i+1,0} - b_{i,0} = \frac{\sqrt{\pi}}{2} h^{\frac{1}{2}} [y_{i+1}^{\left(\frac{1}{2}\right)} + \left(\frac{4}{\pi} - 1\right) y_i^{\left(\frac{1}{2}\right)}] - \frac{\sqrt{\pi}}{12} h^{\frac{3}{2}} [y_{i+1}^{\left(\frac{3}{2}\right)} + \left(5 - \frac{16}{\pi}\right) y_i^{\left(\frac{3}{2}\right)}]$$
$$+ \frac{\sqrt{\pi}}{180} h^{\frac{5}{2}} [y_{k+1}^{\left(\frac{5}{2}\right)} + \left(\frac{96}{\pi} - 31\right) y_i^{\left(\frac{5}{2}\right)}], \quad \text{(xxi)}$$

The coefficients of $s_{n-1}(x)$ are obtained by applying (ii) and (v) to them.

$$b_{n,0} - b_{n-1,0} = \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{1}{2}\right)} + hb_{n-1,1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + h^2 b_{n-1,2} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)}$$
$$+ h^3\, b_{n-1,3}, \quad \text{(xxii)}$$

$$b_{n-1,1} + \frac{4}{3} hb_{n-1,2} + \frac{8}{5} h^2\, b_{n-1,3} = \frac{\sqrt{\pi}}{2} h^{-\frac{1}{2}} [\left(y_n^{\left(\frac{1}{2}\right)} - y_{n-1}^{\left(\frac{1}{2}\right)} \right) - h y_{n-1}^{\left(\frac{3}{2}\right)} - \frac{1}{2} h^2 y_{n-1}^{\left(\frac{5}{2}\right)}], \quad \text{(xxiii)}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$b_{n-1,2}+2h\,b_{n-1,3}=\frac{\sqrt{\pi}}{4}h^{-\frac{1}{2}}[\left(y_n^{\left(\frac{3}{2}\right)}-y_{n-1}^{\left(\frac{3}{2}\right)}\right)-hy_{n-1}^{\left(\frac{5}{2}\right)}]\,, \tag{xxiv}$$

$$b_{n-1,3}=\frac{\sqrt{\pi}}{12}h^{-\frac{1}{2}}(y_n^{\left(\frac{5}{2}\right)}-y_{n-1}^{\left(\frac{5}{2}\right)})\,, \tag{xxv}$$

When we solved four equations, we arrived at:

$$b_{n-1,2}=\frac{\sqrt{\pi}}{12}h^{-\frac{1}{2}}[\left(3y_n^{\left(\frac{3}{2}\right)}-3y_{n-1}^{\left(\frac{3}{2}\right)}\right)-h\left(2y_n^{\left(\frac{5}{2}\right)}+y_{n-1}^{\left(\frac{5}{2}\right)}\right)], \tag{xxvi}$$

$$b_{n-1,1}=\frac{\sqrt{\pi}}{2}h^{-\frac{1}{2}}\left(y_n^{\left(\frac{1}{2}\right)}-y_{n-1}^{\left(\frac{1}{2}\right)}\right)-\frac{\sqrt{\pi}}{6}h^{\frac{1}{2}}\left(2y_n^{\left(\frac{3}{2}\right)}+y_{n-1}^{\left(\frac{3}{2}\right)}\right)+\frac{\sqrt{\pi}}{180}h^{\frac{3}{2}}\left(16y_n^{\left(\frac{5}{2}\right)}-y_{n-1}^{\left(\frac{5}{2}\right)}\right), \tag{xxvii}$$

$$b_{n,0}-b_{n-1,0}=\frac{\sqrt{\pi}}{2}h^{\frac{1}{2}}[y_n^{\left(\frac{1}{2}\right)}+\left(\frac{4}{\pi}-1\right)y_{n-1}^{\left(\frac{1}{2}\right)}]-\frac{\sqrt{\pi}}{12}h^{\frac{3}{2}}[y_n^{\left(\frac{3}{2}\right)}+(5-\frac{16}{\pi})y_{n-1}^{\left(\frac{3}{2}\right)}]$$
$$+\frac{\sqrt{\pi}}{180}h^{\frac{5}{2}}[y_n^{\left(\frac{5}{2}\right)}+(\frac{96}{\pi}-31)y_{n-1}^{\left(\frac{5}{2}\right)}]. \tag{xxviii}$$

Being a non-singular matrix, we can see that each of the mentioned coefficients has its own unique determination.

The proof for Theorem 1 is complete.

3. Convergences analysis and bound errors:

In this section, we demonstrate the following lema and therm.

Lemma 1: let $f\in C^3[0,1]$, $v_{i,1}=2b_{i,0}-2y_i$ and $v_{i+1,1}=2b_{i+1,0}-2y_{i+1}$

Then: $|v_{i+1,1}|\le\frac{5(i+1)}{9}h^3w_3(f;h)\to|v_{i,1}|\le\frac{5i}{9}h^3w_3(f;h)$, $i=0,1,\dots,n-1$

Proof: -

Since $v_{i,1}=2b_{i,0}-2y_i$ and $v_{i+1,1}=2b_{i+1,0}-2y_{i+1}$

for $x_i\le x\le x_{i+1}$ $i=1,2,\dots,n-2$,we have from (xix)

$$b_{i+1,0}-b_{i,0}=\frac{\sqrt{\pi}}{2}h^{\frac{1}{2}}[y_{i+1}^{\left(\frac{1}{2}\right)}+\left(\frac{4}{\pi}-1\right)y_i^{\left(\frac{1}{2}\right)}]-\frac{\sqrt{\pi}}{12}h^{\frac{3}{2}}[y_{i+1}^{\left(\frac{3}{2}\right)}+\left(5-\frac{16}{\pi}\right)y_i^{\left(\frac{3}{2}\right)}]$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$+\frac{\sqrt{\pi}}{180}h^{\frac{5}{2}}\left[y_{k+1}^{\left(\frac{5}{2}\right)}+\left(\frac{96}{\pi}-31\right)y_i^{\left(\frac{5}{2}\right)}\right]$$

$$\frac{1}{2}(2b_{i+1,0}-2y_{i+1}+2y_{i+1})-\frac{1}{2}(2b_{i,0}-2y_i+2y_i)$$

$$=\frac{\sqrt{\pi}}{2}h^{\frac{1}{2}}\left[y_{i+1}^{\left(\frac{1}{2}\right)}+\left(\frac{4}{\pi}-1\right)y_i^{\left(\frac{1}{2}\right)}\right]-\frac{\sqrt{\pi}}{12}h^{\frac{3}{2}}\left[y_{i+1}^{\left(\frac{3}{2}\right)}+\left(5-\frac{16}{\pi}\right)y_i^{\left(\frac{3}{2}\right)}\right]$$

$$+\frac{\sqrt{\pi}}{180}h^{\frac{5}{2}}\left[16y_{k+1}^{\left(\frac{5}{2}\right)}+\left(\frac{96}{\pi}-31\right)y_i^{\left(\frac{5}{2}\right)}\right]$$

$$\frac{v_{i+1,1}-v_{i,1}}{2}=\frac{\sqrt{\pi}}{2}h^{\frac{1}{2}}\left[y_{i+1}^{\left(\frac{1}{2}\right)}+\left(\frac{4}{\pi}-1\right)y_i^{\left(\frac{1}{2}\right)}\right]-\frac{\sqrt{\pi}}{12}h^{\frac{3}{2}}\left[y_{i+1}^{\left(\frac{3}{2}\right)}+\left(5-\frac{16}{\pi}\right)y_i^{\left(\frac{3}{2}\right)}\right]$$

$$+\frac{\sqrt{\pi}}{180}h^{\frac{5}{2}}[16y_{k+1}^{\left(\frac{5}{2}\right)}+\left(\frac{96}{\pi}-31\right)y_i^{\left(\frac{5}{2}\right)}]-y_{i+1}+y_i$$

by using Taylor series expansion for fractional $f(x)\in c^3[0,1]$, about x_i we have

$$|v_{i+1,1}|\le\frac{5(i+1)}{9}h^3w_3(f;h)\rightarrow|v_{i,1}|\le\frac{5i}{9}h^3w_3(f;h)\ ,\ i=0,1,\dots,n-1$$

Theorem 2: let $f\in C^3[0,1]$ and $S_\delta(x)\in S_p(3,3,n)$ be a unique spline function satisfying the condition of Theorem 1. Then

$\left\|s_\delta^{(p)}(x)-f^{(p)}(x)\right\|\le T\,h^{3-p}\,w_3(f;h), p=0\,(\frac{1}{2})3$ and $w_6(f;h)$ denotes the modulus of continuity of $f^{(3)}$. Where

$$T=\begin{cases}\frac{83}{18} & ,\text{when } x_j\le x\le x_{j+1}\\ \frac{83}{18} & ,\text{when } x_j\le x\le x_{j+1}\\ \frac{8}{3} & ,\text{when } x_j\le x\le x_{j+1}\\ \frac{8}{3} & ,\text{when } x_j\le x\le x_{j+1}\\ 1 & ,\text{when } x_j\le x\le x_{j+1}\\ 1 & ,\text{when } x_j\le x\le x_{j+1}\end{cases}\quad,\ j=0,n-1,i \text{ and } p=0(\tfrac{1}{2})\tfrac{5}{2}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$T = \begin{cases} \frac{(122+5j)}{18} & ,\text{when } x_j \leq x \leq x_{j+1}, j = n-1, i \\ \frac{61}{9} & ,\text{when } x_0 \leq x \leq x_1 \end{cases} \quad \text{and } p = 3$$

Proof: - for $x_j \leq x \leq x_{j+1}$ $j = 1,2, \dots, n-2$,we have from (vii)

$$\left|s_k^{(3)}(x) - f^{(3)}(x)\right| = \left|6\, b_{i,3} - f^{(3)}(x)\right|$$

by using Taylor series expansion for fractional $f(x) \in c^3[0,1]$, about x_j we have:

$$y_k = y_{k-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{k-1}^{\left(\frac{1}{2}\right)} + h y'_{k-1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{k-1}^{\left(\frac{3}{2}\right)} + \frac{h^2}{2!} y''_{k-1} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{k-1}^{\left(\frac{5}{2}\right)}$$

$$+ \frac{h^3}{3!} y^{(3)}(\alpha_1), \quad x_{k-1} < \alpha_1 < x_k, \qquad \text{(xxix)}$$

$$y_k^{\left(\frac{1}{2}\right)} = y_{k-1}^{\left(\frac{1}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y'_{k-1} + h y_{k-1}^{\left(\frac{3}{2}\right)} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y''_{k-1} + \frac{h^2}{2!} y_{k-1}^{\left(\frac{5}{2}\right)} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y^{(3)}(\alpha_2),$$

$$x_{k-1} < \alpha_2 < x_k, \qquad \text{(xxx)}$$

$$y'_k = y'_{k-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{k-1}^{\left(\frac{3}{2}\right)} + h y''_{k-1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{k-1}^{\left(\frac{5}{2}\right)} + \frac{h^2}{2!} y^{(3)}(\alpha_3), \quad x_{k-1} < \alpha_3 < x_k, \qquad \text{(xxxi)}$$

$$y_k^{\left(\frac{3}{2}\right)} = y_{k-1}^{\left(\frac{3}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y''_{k-1} + h y_{k-1}^{\left(\frac{5}{2}\right)} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y^{(3)}(\alpha_4), \quad x_{k-1} < \alpha_4 < x_k, \qquad \text{(xxxii)}$$

$$y''_k = y''_{k-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{k-1}^{\left(\frac{5}{2}\right)} + h y^{(3)}(\alpha_5), \quad x_{k-1} < \alpha_5 < x_k, \qquad \text{(xxxiii)}$$

$$y_k^{\left(\frac{5}{2}\right)} = y_{k-1}^{\left(\frac{5}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y^{(3)}(\alpha_6), \quad x_{k-1} < \alpha_6 < x_k, \qquad \text{(xxxiv)}$$

$$y_k^{(3)} = y^{(3)}(\alpha_7) \quad x_{k-1} < \alpha_7 < x_k, \qquad \text{(xxxv)}$$

Putting $= x_i$, $k = i + 1$ in(xxxiv)- (xxxv) and (xviii) we obtain:

$$\left|s_i^{(3)}(x) - f^{(3)}(x)\right| = \left|6\, b_{i,3} - f^{(3)}(x)\right| \leq w_3(f,h)\,, \qquad \text{(xxxvi)}$$

Also putting $= x_{n-1}$, $k = n$ in(xxxiv)- (xxxv) and (xxv) we get:

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$\left|s_{n-1}^{(3)}(x) - f^{(3)}(x)\right| = \left|6\, b_{n-1,3} - f^{(3)}(x)\right| \leq w_3(f,h)\,, \qquad \text{(xxxvii)}$$

and

putting $x = x_0\,, k = 1$ in(xxxiv) − (xxxv)and (xiv)we have:

$$\left|s_0^{(3)}(x) - f^{(3)}(x)\right| = \left|6\, b_{0,3} - f^{(3)}(x)\right| \leq w_3(f,h)\,, \qquad \text{(xxxviii)}$$

Since $s_i^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_i^{(3)}(x) - f^{(3)}(x)) + s_i^{\left(\frac{5}{2}\right)}(x_i) - f^{\left(\frac{5}{2}\right)}(x_i)$

By using (xxxvi) and (i) we get:

$$\left|s_i^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq I_0^{x\ \frac{1}{2}}(\left|s_i^{(3)}(x) - f^{(3)}(x)\right|) \text{since } s_i^{\left(\frac{5}{2}\right)}(x_i) - f^{\left(\frac{5}{2}\right)}(x_i) = 0$$

$$\left|s_i^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq h^{\frac{1}{2}} w_3(f;h), \qquad \text{(xxxix)}$$

By using (xxxvii) and (ii) we obtain:

$$\left|s_{n-1}^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq I_0^{x\ \frac{1}{2}}(\left|s_{n-1}^{(3)}(x) - f^{(3)}(x)\right|) \text{since } s_{n-1}^{\left(\frac{5}{2}\right)}(x_i) - f^{\left(\frac{5}{2}\right)}(x_i) = 0$$

$$\left|s_{n-1}^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq h^{\frac{1}{2}} w_3(f;h), \qquad \text{(xl)}$$

Also using (xxxviii) and (i) we have:

$$\left|s_0^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq I_0^{x\ \frac{1}{2}}(\left|s_0^{(3)}(x) - f^{(3)}(x)\right|) \text{since } s_0^{\left(\frac{5}{2}\right)}(x_i) - f^{\left(\frac{5}{2}\right)}(x_i) = 0$$

$$\left|s_0^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \leq h^{\frac{1}{2}} w_3(f;h), \qquad \text{(xli)}$$

Because $s_i''(x) = 2b_{i,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{5}{2}\right)} + 6h\, b_{i,3}$

$$s_i''(x) - f''(x) = h\left(s_i^{(3)}(x) - f^{(3)}(x)\right) + 2b_{i,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{5}{2}\right)} + hf^{(3)}(x) - f''(x)$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$|s_i''(x)-f''(x)|\leq h\left|s_i^{(3)}(x)-f^{(3)}(x)\right|+\frac{5}{3}hw_3(f;h)$$

Putting $=x_i$, $k=i+1$ in (xxxii)- (xxxv) , (xix) and (xxxix) we get:

$$|s_i''(x)-f''(x)|\leq hw_3(f;h)+\frac{5}{3}hw_3(f;h)$$

$$|s_i''(x)-f''(x)|\leq\frac{8}{3}hw_3(f;h) \qquad \text{(xlii)}$$

also since $s_{n-1}''(x)=2b_{n-1,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_{n-1}^{\left(\frac{5}{2}\right)}+6h\,b_{n-1,3}$

$$s_{n-1}''(x)-f''(x)=h\left(s_{n-1}^{(3)}(x)-f^{(3)}(x)\right)+2b_{n-1,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_{n-1}^{\left(\frac{5}{2}\right)}+hf^{(3)}(x)-f''(x)$$

putting $=x_i$, $k=n$ in (xxxii)- (xxxv), (xxvi) and (xxxvii) we obtain:

$$|s_{n-1}''(x)-f''(x)|\leq hw_3(f;h)+\frac{5}{3}hw_3(f;h)$$

$$|s_{n-1}''(x)-f''(x)|\leq\frac{8}{3}hw_3(f;h) \qquad \text{(xliii)}$$

Now $s_0''(x)=2\,b_{0,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\;y_0^{\left(\frac{5}{2}\right)}+6h\,b_{0,3}$

$$s_0''(x)-f''(x)=2\,b_{0,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\;y_0^{\left(\frac{5}{2}\right)}+6h\,b_{0,3}-f''(x)$$

$$s_0''(x)-f''(x)=h\left(s_0^{(3)}(x)-f^{(3)}(x)\right)+2b_{0,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\,y_0^{\left(\frac{5}{2}\right)}+hf^{(3)}(x)-f''(x)$$

putting $=x_i$, $k=1$ in (xxxii)- (xxxv), (xiii) and (xxxviii) we have that:

$$|s_0''(x)-f''(x)|\leq\frac{8}{3}hw_3(f;h)\,, \qquad \text{(xliv)}$$

because $s_i^{\left(\frac{3}{2}\right)}(x)-f^{\left(\frac{3}{2}\right)}(x)=I_0^{x\;\frac{1}{2}}(s_i''(t)-f''(t)+s_i^{\left(\frac{3}{2}\right)}(x_i)-f^{\left(\frac{3}{2}\right)}(x_i)$

$s_i^{\left(\frac{3}{2}\right)}(x)-f^{\left(\frac{3}{2}\right)}(x)=I_0^{x\;\frac{1}{2}}(s_i''(t)-f''(t))$ since $s_i^{\left(\frac{3}{2}\right)}(x_i)=f^{\left(\frac{3}{2}\right)}(x_i)$

Using (xlii) and (i) we get:

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$\left|s_i^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x)\right| \leq \frac{8}{3} h^{\frac{3}{2}} w_3(f;h) \,, \qquad \text{(xlv)}$$

Because $s_{n-1}^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x) = I_0^x\ ^{\frac{1}{2}}(s''_{n-1}(t) - f''(t) + s_{n-1}^{\left(\frac{3}{2}\right)}(x_i) - f^{\left(\frac{3}{2}\right)}(x_i)$

$s_{n-1}^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x) = I_0^x\ ^{\frac{1}{2}}(s''_{n-1}(t) - f''(t))$ since $s_{n-1}^{\left(\frac{3}{2}\right)}(x_i) = f^{\left(\frac{3}{2}\right)}(x_i)$

Using (xliii)and (ii) we get:

$$\left|s_{n-1}^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x)\right| \leq \frac{8}{3} h^{\frac{3}{2}} w_3(f;h) \,, \qquad \text{(xlvi)}$$

Now $s_0^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x) = I_0^x\ ^{\frac{1}{2}}(s''_0(t) - f''(t) + s_0^{\left(\frac{3}{2}\right)}(x_i) - f^{\left(\frac{3}{2}\right)}(x_i)$

$s_0^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x) = I_0^x\ ^{\frac{1}{2}}(s''_0(t) - f''(t))$ since $s_0^{\left(\frac{3}{2}\right)}(x_i) = f^{\left(\frac{3}{2}\right)}(x_i)$

Using (xliv) and (i) we get:

$$\left|s_0^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x)\right| \leq \frac{8}{3} h^{\frac{3}{2}} w_3(f;h) \,, \qquad \text{(xlvii)}$$

Since

$$s'_i(x) - f'(x) = b_{i,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{3}{2}\right)} + 2h b_{i,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_i^{\left(\frac{5}{2}\right)} + 3h^2 b_{i,3} - f'(x)$$

$$s'_i(x) - f'(x) = b_{i,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{3}{2}\right)} + h[2\, b_{i,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}}\ y_i^{\left(\frac{5}{2}\right)} + 6h\, b_{i,3} - f''(x)]$$

$$+ \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_i^{\left(\frac{5}{2}\right)} - f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}}\, y_i^{\left(\frac{5}{2}\right)} - \frac{h^2}{2}[6\, b_{i,3} - f^{(3)}(x)]$$

$$-\frac{h^2}{2} f^{(3)}(x) + hf''(x)$$

Putting $= x_i$, $k = i + 1$ in (xxxi)- (xxxv) , (xlii) and (xxxvi) we obtain:

$$|s'_i(x) - f'(x)| \leq \frac{83}{18} h^2 w_3(f;h), \qquad \text{(xlviii)}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

Because $s'_{n-1}(x) = b_{n-1,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + 2hb_{n-1,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} + 3h^2 b_{n-1,3}$

$$s'_{n-1}(x) - f'(x) = b_{n-1,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + 2hb_{n-1,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} + 3h^2 b_{n-1,3} - f'(x)$$

$$s'_{n-1}(x) - f'(x) = b_{n-1,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + h[2\, b_{n-1,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}}\ y_i^{\left(\frac{5}{2}\right)} + 6h\, b_{n-1,3} - f''(x)]$$

$$+ \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} - f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}}\ y_{n-1}^{\left(\frac{5}{2}\right)} - \frac{h^2}{2}[6\, b_{n-1,3} - f^{(3)}(x)]$$

$$-\frac{h^2}{2} f^{(3)}(x) + hf''(x)$$

Putting $= x_i$, $k = n$ in (xxxi)- (xxxv) , (xliv) and (xxxvii) we get:

$$|s'_{n-1}(x) - f'(x)| \le \frac{83}{18} h^2 w_3(f;h), \qquad \text{(xlix)}$$

If $s'_0(x) = b_{0,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_0^{\left(\frac{3}{2}\right)} + 2hb_{0,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_0^{\left(\frac{5}{2}\right)} + 3h^2 b_{0,3}$

$$s'_0(x) - f'(x) = b_{0,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_0^{\left(\frac{3}{2}\right)} + 2hb_{0,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_0^{\left(\frac{5}{2}\right)} + 3h^2 b_{0,3} - f'(x)$$

$$s'_0(x) - f'(x) = b_{0,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_0^{\left(\frac{3}{2}\right)} + 2hb_{0,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_0^{\left(\frac{5}{2}\right)} + 3h^2 b_{0,3} - f'(x)$$

$$s'_0(x) - f'(x) = b_{0,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_0^{\left(\frac{3}{2}\right)} + h[2\, b_{0,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}}\ y_0^{\left(\frac{5}{2}\right)} + 6h\, b_{0,3} - f''(x)]$$

$$+ \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_0^{\left(\frac{5}{2}\right)} - f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}}\ y_0^{\left(\frac{5}{2}\right)} - \frac{h^2}{2}[6\, b_{0,3} - f^{(3)}(x)]$$

$$-\frac{h^2}{2} f^{(3)}(x) + hf''(x)$$

$$s'_0(x) - f'(x) = b_{0,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_0^{\left(\frac{3}{2}\right)} + h[s''_0(x) - f''(x)] - \frac{h^2}{2}[s_0^{(3)}(x) - f^{(3)}(x)]$$

$$+ \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_0^{\left(\frac{5}{2}\right)} - f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}}\ y_0^{\left(\frac{5}{2}\right)} - \frac{h^2}{2} f^{(3)}(x) + hf''(x)$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

Putting $= x_i$, $k = 1$ in (xxxi)- (xxxv) , (xliii) and (xxxviii)we have:

$$|s_0'(x) - f'(x)| \leq \frac{83}{18} h^2 w_3(f;h) \quad , \tag{l}$$

Since $s_i^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_i'(t) - f'(t) + s_i^{\left(\frac{1}{2}\right)}(x_i) - f^{\left(\frac{1}{2}\right)}(x_i)$

$s_i^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_i'(t) - f'(t))$ since $s_i^{\left(\frac{1}{2}\right)}(x_i) = f^{\left(\frac{1}{2}\right)}(x_i)$

By (i) and (xlviii)

$$\left|s_i^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x)\right| \leq \frac{83}{18} h^{\frac{5}{2}} w_3(f;h) \quad , \tag{li}$$

Because $s_{n-1}^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_{n-1}'(t) - f'(t) + s_{n-1}^{\left(\frac{1}{2}\right)}(x_i) - f^{\left(\frac{1}{2}\right)}(x_i)$

$s_{n-1}^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_{n-1}'(t) - f'(t))$ since $s_{n-1}^{\left(\frac{1}{2}\right)}(x_i) = f^{\left(\frac{1}{2}\right)}(x_i)$

By (ii) and (xlix)

$$\left|s_{n-1}^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x)\right| \leq \frac{83}{18} h^{\frac{5}{2}} w_3(f;h), \tag{lii}$$

now $s_0^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_0'(t) - f'(t) + s_0^{\left(\frac{1}{2}\right)}(x_i) - f^{\left(\frac{1}{2}\right)}(x_i)$

by (i) and (l)

$s_0^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x) = I_0^{x\ \frac{1}{2}}(s_0'(t) - f'(t))$ since $s_0^{\left(\frac{1}{2}\right)}(x_i) = f^{\left(\frac{1}{2}\right)}(x_i)$

$$\left|s_0^{\left(\frac{1}{2}\right)}(x) - f^{\left(\frac{1}{2}\right)}(x)\right| \leq \frac{83}{18} h^{\frac{5}{2}} w_3(f;h), \tag{liii}$$

Because

$$s_i(x) = b_{i,0} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_i^{\left(\frac{1}{2}\right)} + h b_{i,1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_i^{\left(\frac{3}{2}\right)} + h^2 b_{i,2} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_i^{\left(\frac{5}{2}\right)} + h^3\, b_{i,3}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$s_i(x)-f(x)=b_{i,0}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{1}{2}\right)}+hb_{i,1}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}+h^2b_{i,2}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}$$

$$+h^3\,b_{i,3}-f(x)$$

$$s_i(x)-f(x)=b_{i,0}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{1}{2}\right)}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}-f(x)+hf'(x)$$

$$+h[b_{i,1}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{3}{2}\right)}+2hb_{i,2}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{5}{2}\right)}+3h^2b_{i,3}-f'(x)]$$

$$-\frac{2}{\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}-\frac{4}{3\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}[2b_{i,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{5}{2}\right)}+6h\,b_{i,3}-f''(x)]$$

$$+\frac{1}{\sqrt{\pi}}h^{\frac{5}{2}}\;y_i^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}f''(x)+\frac{h^3}{6}[6\,b_{i,3}-f^{(3)}(x)]+\frac{h^3}{6}f^{(3)}(x)$$

$$s_i(x)-f(x)=h[s_i'(x)-f'(x)]-\frac{h^2}{2}[s_i''(x)-f''(x)]+\frac{h^3}{6}[s_i^{(3)}(x)-f^{(3)}(x)]$$

$$+\frac{1}{2}[2b_{i,0}-2y_i]+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{1}{2}\right)}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}-f(x)+hf'(x)$$

$$-\frac{2}{\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}-\frac{4}{3\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}+\frac{1}{\sqrt{\pi}}h^{\frac{5}{2}}\;y_i^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}f''(x)+\frac{h^3}{6}f^{(3)}(x)+y_i$$

$$s_i(x)-f(x)=h[s_i'(x)-f'(x)]-\frac{h^2}{2}[s_i''(x)-f''(x)]+\frac{h^3}{6}[s_i^{(3)}(x)-f^{(3)}(x)]$$

$$+\frac{1}{2}v_{i,1}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{\left(\frac{1}{2}\right)}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}-f(x)+hf'(x)$$

$$-\frac{2}{\sqrt{\pi}}h^{\frac{3}{2}}y_i^{\left(\frac{3}{2}\right)}-\frac{4}{3\sqrt{\pi}}h^{\frac{5}{2}}y_i^{\left(\frac{5}{2}\right)}+\frac{1}{\sqrt{\pi}}h^{\frac{5}{2}}\;y_i^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}f''(x)+\frac{h^3}{6}f^{(3)}(x)+y_i$$

Putting $=x_i$, $k=i+1$ in (xxix)- (xxxv), lemma1, (xlviii), (xlii) and (xxxvi) we get:

$$|s_i(x)-f(x)|\leq\frac{(122+5i)}{18}h^3w_3(f;h)\ ,$$

Since $s_{n-1}(x_n)=b_{n-1,0}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_{n-1}^{\left(\frac{1}{2}\right)}+hb_{n-1,1}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_{n-1}^{\left(\frac{3}{2}\right)}+h^2b_{n-1,2}$

$$+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_{n-1}^{\left(\frac{5}{2}\right)}+h^3\,b_{n-1,3}$$

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$s_{n-1}(x_n) - f(x) = b_{n-1,0} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{1}{2}\right)} + h b_{n-1,1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + h^2 b_{n-1,2}$$

$$+ \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} + h^3\, b_{n-1,3} - f(x)$$

$$s_{n-1}(x) - f(x) = b_{n-1,0} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{1}{2}\right)} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} - f(x) + h f'(x)$$

$$+ h\left[b_{n-1,1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} + 2h b_{n-1,2} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} + 3h^2 b_{n-1,3} - f'(x)\right]$$

$$- \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} - \frac{4}{3\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} - \frac{h^2}{2}\left[2b_{n-1,2} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} + 6h\, b_{n-1,3} - f''(x)\right]$$

$$+ \frac{1}{\sqrt{\pi}} h^{\frac{5}{2}}\; y_{n-1}^{\left(\frac{5}{2}\right)} - \frac{h^2}{2} f''(x) + \frac{h^3}{6}\left[6\, b_{n-1,3} - f^{(3)}(x)\right] + \frac{h^3}{6} f^{(3)}(x)$$

$$s_{n-1}(x) - f(x) = h\left[s'_{n-1}(x) - f'(x)\right] - \frac{h^2}{2}\left[s''_{n-1}(x) - f''(x)\right] + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{1}{2}\right)}$$

$$+ \frac{h^3}{6}\left[s_{n-1}^{(3)}(x) - f^{(3)}(x)\right] + \frac{1}{2}\left[2b_{n-1,0} - 2y_{n-1}\right] + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)}$$

$$+ \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} - f(x) + h f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} - \frac{4}{3\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)}$$

$$+ \frac{1}{\sqrt{\pi}} h^{\frac{5}{2}}\; y_{n-1}^{\left(\frac{5}{2}\right)} - \frac{h^2}{2} f''(x) + \frac{h^3}{6} f^{(3)}(x) + y_{n-1}$$

$$s_{n-1}(x) - f(x) = h\left[s'_{n-1}(x) - f'(x)\right] - \frac{h^2}{2}\left[s''_{n-1}(x) - f''(x)\right] + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{n-1}^{\left(\frac{1}{2}\right)}$$

$$+ \frac{h^3}{6}\left[s_{n-1}^{(3)}(x) - f^{(3)}(x)\right] + \frac{1}{2} v_{n-1,1} + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} - \frac{h^2}{2} f''(x)$$

$$+ \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)} - f(x) + h f'(x) - \frac{2}{\sqrt{\pi}} h^{\frac{3}{2}} y_{n-1}^{\left(\frac{3}{2}\right)} - \frac{4}{3\sqrt{\pi}} h^{\frac{5}{2}} y_{n-1}^{\left(\frac{5}{2}\right)}$$

$$+ \frac{1}{\sqrt{\pi}} h^{\frac{5}{2}}\; y_{n-1}^{\left(\frac{5}{2}\right)} + \frac{h^3}{6} f^{(3)}(x) + y_{n-1}$$

Putting $= x_i$, $k = n$ in (xxix)- (xxxv), lemma1, (xxxvii), (xliii) and (xlix) we get:

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

$$|s_{n-1}(x)-f(x)| \le \frac{[122+5(n-1)]}{18} h^3 w_3(f;h)$$

Let $s_0(x)=y_0+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_0^{\left(\frac{1}{2}\right)}+hb_{0,1}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}+h^2b_{0,2}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}+h^3\,b_{0,3}$

$$s_0(x)-f(x)=y_0+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_0^{\left(\frac{1}{2}\right)}+hb_{0,1}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}+h^2b_{0,2}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}$$

$$+h^3\,b_{0,3}-f(x)$$

$$s_0(x)-f(x)=y_0+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_0^{\left(\frac{1}{2}\right)}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}-f(x)+hf'(x)$$

$$+h[b_{0,1}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_0^{\left(\frac{3}{2}\right)}+2hb_{0,2}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{5}{2}\right)}+3h^2b_{0,3}-f'(x)]$$

$$-\frac{2}{\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}-\frac{4}{3\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}[2\,b_{0,2}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\;y_0^{\left(\frac{5}{2}\right)}+6h\,b_{0,3}-f''(x)]$$

$$+\frac{1}{\sqrt{\pi}}h^{\frac{5}{2}}\;y_0^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}f''(x)+\frac{h^3}{6}[6\,b_{0,3}-f^{(3)}(x)]+\frac{h^3}{6}f^{(3)}(x)$$

$$s_0(x)-f(x)=h[s_0'(x)-f'(x)]-\frac{h^2}{2}[s_0''(x)-f''(x)]+\frac{h^3}{6}[s_0^{(3)}(x)-f^{(3)}(x)]$$

$$+y_0+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_0^{\left(\frac{1}{2}\right)}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}+\frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}-f(x)+hf'(x)$$

$$-\frac{2}{\sqrt{\pi}}h^{\frac{3}{2}}y_0^{\left(\frac{3}{2}\right)}-\frac{4}{3\sqrt{\pi}}h^{\frac{5}{2}}y_0^{\left(\frac{5}{2}\right)}+\frac{1}{\sqrt{\pi}}h^{\frac{5}{2}}\;y_0^{\left(\frac{5}{2}\right)}-\frac{h^2}{2}f''(x)+\frac{h^3}{6}f^{(3)}(x)$$

Putting $=x_i$, $k=1$ in (xxix)- (xxxv), (xxxviii), (xliv) and (l) we get:

$$|s_0(x)-f(x)| \le \frac{61}{9} h^3 w_3(f;h)$$

This concludes **Theorem 2's** proof.

3. Conclusion

The purpose of the research described in this article was to construct and evaluate the lacunary spline function. The Caputo fractional derivative for the polynomial spline approach served as the foundation for the proposed scheme. The study's findings revealed the accuracy and efficiency of

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

the devised scheme in determining these kinds of boundary convergences. Additionally, the method explores the intricacies of the fractional polynomial spline, a novel approach that combines to determine fractional spline. The process is meticulously designed to manage fractional situations for fractional polynomial splines, and an examination of truncation mistakes played a crucial role in its creation. Researchers and professionals in the domains of fractional calculus and numerical analysis may find value in the study's findings.

References

[1] R. G. Karem , K. H F. Jwamer , F. K. Hamasalh, Lacunary interpolation by the Spline Function of Fractional Degree, J. Al-Qadisiyah Comput. Sci. Math., 15(2023)54–72. https://doi.org/10.29304/jqcm.2023.15.3.1282.

[2] F. K. Hamasalh, A. H. Ali, On the Optimality of Generalized Fractional Cubic Spline with Applications, AIP Conference Proceedings 2096 (2019) 020004, https://doi.org/10.1063/1.5097801

[3] P. O. Mohammed, F. k. Hamasalh, Th. Abdeljawad, Difference monotonicity analysis on discrete fractional operators with discrete generalized Mittag-Leffler kernels. Springer, Adv. in Diff. Eq., 2021(2021) 213 https://doi.org/10.1186/s13662-021-03372-2.

[4] F. K. Hamasalh, S. S. Hamasalh, Spline Fractional Polynomial for Computing Fractional Differential Fractional Differential Equations, J. Univ. Babylon Pure App. Sci. 30(2)2022. https://doi.org/10.29196/jubpas.v30i2.4185.

[5] F. K. Hamasalh, Fractional Polynomial Spline for solving Differential Equations of Fractional Order, Math. Sci. Letters an International J., 4(2015) 291- 296. http://dx.doi.org/10.12785/msl/040312.

[6] F. K. Hamasalh , S. S. Hamasalh, Comparative Numerical Solution of Fractional Spline with Continuity Equations, Tikrit J. Pure Sci.,28(2023)112–117, https://doi.org/10.25130/tjps.v28i2.1344.

[7] K. H-F Jwamer, F. K. Hamasalh, and R. K. G, Approximation new error bounded by spline degree six, acta universitatis apulensis, Special Issue (2001)201-210. https://wwwemis.de › journals

[8] R. K. Saeed. and K. H-F Jwamer, Minimizing Error Bounds in Lacunary interpolation by

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

spline function (0, 1, 4) case , J. Al-Nahrain univ.,8(2005), 114-119.
https://anjs.edu.iq/index.php/anjs/article/view/1640/1459.

[9] K. H-F Jwamer, On Sixtic Lacunary spline solutions of second order initial value problem, J. of math. and stat., new work,5(2009)369-374. https://doi.org/10.3844/jmssp.2009.369.374.

[10] K. H-F Jwamer and F. K. Hama-Salh , Minimizing error bounds in lacunary interpolation by spline (0,2) Case, J. of Kirkuk Univ, .4(2009),117-124. https://doi.org/10.32894/kujss.2009.41661.

[11] A.Y. Al-Bayati , R. K. Saeed, K. H-F Jwamer, The Existence, Uniqueness and Upper Bounds for Errors of six-degree spline interpolating the lacunary that (0, 2, 5), Al-Rafiden J. Computer Sci. and Math. ,7(2010) 49-57. https://doi.org/10.33899/csmj.2010.163884.

[12] V. Daftardar-Gejji, Fractional Calculus and Fractional Differential Equations; Springer, Gateway East,2019, https://doi.org/10.1007/978-981-13-9227-6.

[13] A. Kammanee, Numerical Solution of Fractional Differential Equations with variable Coefficients by Taylor Basis Functions, Kyungpook Math. J.,61(2021) 383-393. https://doi.org/10.5666/KMJ.2021.61.2.383.

[14] G. Yang ,The property of a special type of exponential spline function. Adv. in Pure Math., 13(2015) 804–807, https://doi.org/10.4236/apm.2015.513074.

[15] R. G. Karem , K. H. F. Jwamer , F. K. Hamasalh, Existence, Uniqueness, and Stability Results for Fractional Differential Equations with Lacunary Interpolation by the Spline Method, Math. and Stati. ,4(2023) 669-675, https://doi.org/10.13189/ms.2023.110408.

[16] Kh. Maleknejad, J. Rashidinia, H. Jalilian, Non-polynomial spline functions and quasi-linearization to approximate nonlinear Volterra integral equation. Faculty of Sci. and Math., Univ. of Niˇs, Serbia,11(2018)3947–3956, https://doi.org/10.2298/fil1811947m .

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).

التحقيق في الاستيفاء الجوفي القائم على وظيفة الشريحة الكسرية مع تحليل التقارب

رضا غفور كريم *، كاروان حمةفرج جوامير و فريدون قادر حمةصالح

جامعة السليمانية , كلية تربية الاساسية, قسم الرياضيات, السليمانية , العراق.

جامعة السليمانية , كلية العلوم , قسم الرياضيات, السليمانية , العراق.

جامعة السليمانية التقنية, معهد بكرجو التقني, السليمانية , العراق.

المستخلص

في هذه الدراسة ، تم اشتقاق درجة كسرية (2/1 ، 2/3 ، 2/5) تقنية الاستيفاء الجوفي. نعرض كيفية دقة الوظائف المعقدة وتطورها في الفاصل الزمني المحدد باستخدام وظيفة ثلاثية الأشرطة. يتم استخدام تقنية الشريحة الممتدة المعدلة المحسنة للحصول على أنواع متعددة من الشروط الحدودية ، بما في ذلك سطوع الترتيب الكسري. يظهر تأثير المشتق الكسري على النموذج من خلال تحليل التقارب الذي تتم محاكاته لقيم مختلفة من بيتا. يمكن حل المشكلات التي تتضمن وظائف الشريحة بمساعدة تقنية الاستيفاء هذه لإنشاء وظائف الشريحة.

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).