Applying Kamal Transformation Method to Solve Fractional Order T. Regge Problem

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ABSTRACT

The aim of this work is to solve the fractional order T. Regge problem by applying a unique transformation technique called the Kamal transform. It focuses mostly on the application of Kamal's transformation methodology to the solution of fractional differential equations (FDEs), especially with regard to the boundary value fractional order T. Regge problem. Furthermore, Kamal transformation formulas for fractional derivatives and fractional integrals are derived, the benefits and drawbacks of the methodology are examined, and multiple instances are shown to demonstrate the practicality of the technique. As seen by the examples we presented, the Kamal transformation methodology provides guaranteed solutions for FDEs, which have a substantial impact on the field of fractional computation.

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1. Introduction

The understanding of characteristic transmission and retention in materials and processes achieving the application of fractional calculus [1, 2]. Numerous scientific and technological domains, including as biology, chemistry, viscoelasticity, anomalous diffusion, fluid mechanics, acoustics, and control theory, use this technique. In these areas of mathematics, fractional differential equations are involved, particularly integro-differential equations with singularities [3-6]. Researchers have devised analytical or numerical approaches to solve these fractional differential equations Many transformation can solve many fractional differential equations such as Rishi transform [7, 8] Kamal transformation[9] Sawi transform [10, 11] Sumudu transform [12]. The solution to the fractional order T.Regge problem is covered in this article. Previous works [13-15] have proven this solution's existence and uniqueness. This study focuses on solving the T.Regge problem for fractional order, using the Rishi transformation approach when applicable and meeting the requirements for the original function [13-15]. Previous study was suggested several numerical or analytical techniques for solving fractional differential equations, including as [5, 16-18]. In this research, we studied solutions for the fractional T. Regge boundary value problem, as is known [19], Studying the Schrodinger operation on the half-axis $R+$ with potential $q(t)$ compactly supported on this interval is related to studying the Regge spectral issue on this interval. $[0, a]$. This problem defined as the form

$$-u''(x)+q(x)u(x)=\lambda^2 p(x)u(x) \quad ; \quad x\in[0,a],$$
$$U_1(u)=u(0)=0 \quad , \quad U_2(u)=u'(a)-i\lambda u(a)=0 \ , \tag{1}$$

And the T.Regge problem for fractional order in [13], [14] defined as:

$$-{}_0^C D_t^{\alpha} u(t)+q(t)\ u(x)=\lambda^2 p(t)\ u(t); \qquad x\in[0,a], \qquad 1<\alpha\leq 2 \tag{2}$$

And the same boundary conditions known as Regge conditions such that the unknown function $u(x)\in C[0,a]$, and the variable coefficients $q(t),\ \ p(t)\in L_+\ [0,a]$.

λ Is a parameter in the spectrum.[14], [19], [20]. We introduced the uniqueness and existence theorems for fractional order ordinary differential equations [13], [15], [21]. In order to solve the fractional order Regge issue, the Kamal transformation is created in this study. The following is how the paper is set up: The essential concepts and characteristics of fractional calculus are provided in Section 2. The fractional integral and derivative Kamal Transform is presented in Section 3. Derived solution of T.Regge fractional order via Kamal transform has been showed in section 4.To demonstrate the efficacy of the suggested strategy, Section 5 provides a few instances of FDEs.

2. Preliminaries

Fractional integrals, derivatives, and other integral transforms are used in both pure and applied mathematics to solve a wide range of differential and integral equations. Wherein we make use of crucial definitions for our research, like: first, Fractional Integral of Order α [5], [16], [22]: for each α > 0 and a integrable function $y(x)$, the right FI of order α is defined:

$$ {}_aI_t^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-s)^{\alpha-1} y(s) ds, \quad -\infty \le a < x < \infty \tag{3}$$

Second, [16], [23] for every α, and $L = \lceil \alpha \rceil$ the Riemann-Liouville derivative of order α defined as

$$ {}_aD_t^\alpha y(x) = \frac{1}{\Gamma(L-\alpha)} \frac{d^L}{dx^L} \int_a^t (x-s)^{L-\alpha-1} y(s) ds \tag{4}$$

Third, let $\alpha > 0, L = \lceil \alpha \rceil$. The Caputo derivative operator of order α and $y(x)$ be n-times differentiable function, $x > a$ is defined as [5], [16]

$$ {}_a^CD_t^\alpha y(x) = \frac{1}{\Gamma(L-\alpha)} \int_a^t (x-j)^{L-\alpha-1} \left(\frac{d}{dj}\right)^L y(j) dj \tag{5}$$

Remark 1: [2], [10], [16], [21], [22], [24] The following are some fundamental characteristics of fractional calculus:

1. 1. A fractional operator is a linear operator (integral and differential).
2. The following is the definition of composition between two Riemann-Liouville integrations of orders e and b:

$$ {}_aI_s^e \, {}_aI_s^b f(s) = {}_aI_s^b \, {}_aI_s^e f(s) = I_t^{e+b} f(s). \tag{6}$$

3. For $l \ge \beta$, and $f(s) \in C[a,b]$, and for every element $s \in [a,b]$, then

$$ {}_{a}^{RL}D_s^{l}\left({}_{a}I_s^{\beta} f(t)\right) = {}_{a}^{RL}D_s^{l-\beta} f(s). \quad (7)$$

The relation is done.

4. The definition of composition of the Liouville-Caputo operator of order α between fractional (differentiation and integration) is as follows:

$$ {}_{a}^{LC}D_s^{\beta}\left({}_{a}I_s^{\beta} f(s)\right) = f(s). \quad (8)$$

5. Liouville-Caputo operator of order α: between fractional (integration and differentiation) composition, and $m = \lceil \alpha \rceil$ is defined as:

$$ {}_{a}I_x^{\alpha}\left({}_{a}^{LC}D_x^{k} f(x)\right) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^k}{k!} f^{(k)}(a). \quad (9)$$

In general, ${}_{a}^{LC}D_x^{k}\left({}_{a}I_x^{\alpha} f(x)\right) \neq {}_{a}I_x^{\alpha}\left({}_{a}^{LC}D_x^{k} f(x)\right)$.

6. Use the differential and fractional integral of the Liouville-Caputo of function $s^m, m \geq 0$, we obtain : ${}_{a}I_t^{\alpha} s^m = \frac{\Gamma(1+m)}{\Gamma(1+m+\alpha)} s^{n+\alpha}$ and ${}_{a}^{LC}D_t^{k} s^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)} s^{m-\alpha}$. (10)

3. The Kamal Transformation [9]

Based on the fundamental properties and simplicity of the mathematics involved, the Kamal Transform derives its name from the classic Fourier integral. The Kamal transform was created by Abdelilah Kamal to facilitate the time domain solution of ordinary and partial differential equations. Mathematical solutions for differential equations usually involve the use of the Fourier transformation, Laplace, Sumudu, and Elzaki, Aboodh, and Mahgoub transforms. Moreover, Kamal transform and some of its basic ingredients are employed.

Kamal Transform of the function $f(t)$ for $t \geq 0$ is defined by the integral

$$K[f(t)] = F(v) = \int_0^{\infty} f(t)\; e^{\frac{-t}{v}} dt. \qquad t \geq 0,\; k_1 \leq v \leq k_2,$$

and it is denoted by the operator $K(.)$

If $K[f(t)] = F(v)$, then $f\,(t)$ is the inverse Kamal Transform of $F(v)$. in symbol,

$$f(t) = K^{-1}\ [F(v)] = K^{-1}\left[\int_0^{\infty} f(t)\ e^{\frac{-t}{v}} dt\right]$$

Where K^{-1} is the inverse Kamal Transform operator.

Properties 1:[25]Kamal Transformation for some known function for $t > 0, v \in \mathbb{C}, a \in R, n \in N$:

i. $K[1] = v$
ii. $K\left[x^{\partial}\right] = \partial!\, v^{\partial} = v^{\partial+1}\Gamma(\partial+1), \partial \geq 0$
iii. $K[e^{at}] = \frac{v}{1-av}$
iv. $K[sin\ at] = \frac{av^2}{1+a^2v^2}$
v. $K[cos\ at] = \frac{v}{1+a^2v^2}$
vi. $K[sinh\ at] = \frac{av^2}{1-a^2v^2}$
vii. $K[cosh\ at] = \frac{v}{1-a^2v^2}$.

Property 2:[25] Let $n \geq 1$ and $G(v)$ be the Kamal Transform of the function $g(t)$. The Kamal Transform of n^{th} derivative of $f(t)$ is given by

$$K\left\{g^{(n)}(t)\right\} = \frac{1}{v^n}G(v) - \frac{1}{v^{n-1}}g(0) - \frac{1}{v^{n-2}}f'(0) \dots \dots - g^{(n-1)}(0),$$

$$K[g^n(t)] = \frac{1}{v^n}G(v) - \sum_{k=0}^{n-1} v^{k-n+1}g^k(0).$$

Property 3:[25] Let $M(v)$ and $N(v)$ denote the Kamal Transform of $m(t)$ and $n(t)$ respectively. If $(m * n)\ (t) = \int_0^t m(\tau)\ n(t-\tau)d\tau$.

Where * denotes convolution of g and g, then the Kamal Transform of the convolution of $m(t)$ and $n(t)$ is $K[m(t) * n(t)] = M(v)\ N(v)$.

Properties 4. Some Properties of the Inverse Kamal Transformation[25]

$G(v)$	$g(t) = K^{-1}\{G(v)\}$	$G(v)$	$g(t) = K^{-1}\{G(v)\}$
v	1	$\frac{v^2}{1+a^2v^2}$	$\frac{sinat}{a}$
v^3	$\frac{t^2}{2!}$	$\frac{v}{1+a^2v^2}$	$cosat$
$v^{n+1}, n \geq 0$	$\frac{t^n}{n!}$	$\frac{v^2}{1-a^2v^2}$	$\frac{sinhat}{a}$
$\frac{v}{1-av}$	e^{at}	$\frac{v^2}{1-a^2v^2}$	$coshat$

3.1 Kamal Transform of Fractional Integrals and Derivatives [26]

3.1.1 Fractional Integral

Proposition 1: If $\partial \in [n-1, n)$, Using the Kamal transform of the fractional integral formula, we have:

$$K\left[I^{\partial} g(t)\right] = K\left[{}^{RL}D_t^{-\partial} g(t)\right] = \frac{1}{\Gamma(\partial)} \int_a^t (t-s)^{\partial-1} g(s)ds = \frac{1}{\Gamma(\partial)} L\left[t^{\partial-1}\right] L[g(t)] = v^{\partial} G(v).$$

3.1.2 Fractional Derivatives

Proposition 2: Let $G(v)$ is a Kamal transform for $g(t)$, the Kamal transform for Reiman-Liouvile fractional Derivative of order α is

$$K[D^{\alpha} g(t)] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} g^{\alpha-k-1}(0).$$

Proof: Reiman-Liouvile fractional derivative and related fractional integral give us

$$K[D^{\alpha} g(t)] = k(D^{n} I^{n-\alpha} g(t)\}$$

And from properties Kamal Transform for derivative we have

$K[g^{n}(t)] = \frac{1}{v^{n}} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} g^{k}(0)$

So $K[D^{\alpha} g(t)] = K\{D^{n} I^{n-\alpha} g(t)\} = \frac{1}{v^{n}}\{K\{I^{n-\alpha} g(t)\} - \sum_{k=0}^{n-1} v^{k-n+1} \frac{d^{n-k-1}}{dt^{n-k-1}} I^{n-\alpha} g(0) =$ $v^{-n} v^{n-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} D^{n-k-1} D^{-(n-\alpha)} g(0) = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} D^{\alpha-k-1} g(0).$

Preposition 3: Let $G(v)$ is a Kamal transform for $g(t)$, the Kamal transform formula for Caputo fractional Derivative of order α is

$K[\,{}^{c}D_{t}^{\alpha} g(t)] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} g^{k}(0),$

Proof: Reiman-Liouvile fractional derivative and related fractional integral give us

$K[D^{\alpha} g(t)] = K(I^{n-\alpha} D^{n} g(t)\}$

So $K[\,{}^{c}D_{t}^{\alpha} g(t)] = K(I^{n-\alpha} D^{n} g(t)\} = v^{n-\alpha}\{K\{D^{n} g(t)\} = v^{n-\alpha}\left\{\frac{1}{v^{n}} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} g^{k}(0)\right\} = v^{n-\alpha} v^{-n} - \sum_{k=0}^{n-1} v^{n-\alpha} v^{k-n+1} g^{k}(0) = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} g^{k}(0).$

4. Solution of fractional order T.Regge boundary value problem by Kamal Transformation Method

In this section we used the Kamal Transformation to find solution of our problem in the following cases. The T. Regge problem for fractional order is defined as

$$-{}_{0}^{C}D_{t}^{\alpha} u(t) + q(t)\ u(t) = \lambda^{2} p(t) u(t) \quad ; \qquad t \in [0, a],\ 1 < \alpha \le 2$$
$$u(0) = 0, \qquad u'(a) - i\lambda u(a) = 0.$$

We solve the T.Regge problem for Farctional order in this section and method by Kamal transformation if it is exist and the conditions of origin function are hold.

Case 1: Constant Coefficients

In this case we suppose that $p(x) = M = q(x)$

The Fractional differential equation is $-{}_0^C D_x^\alpha u(x) + Mu(x) = \lambda^2 Mu(x)$

$\rightarrow {}_0^C D_x^\alpha u(x) + M(\lambda^2 - 1)u(x) = 0$

Apply Kamal transformation for both sides

$K\{{}_0^C D_x^\alpha u(x)\} + \quad K\{M(\lambda^2 - 1)u(x)\} \quad = K\{0\}$

And $K\{u(x)\} = \int_0^\infty e^{-\frac{t}{v}} u(t)dt = U(v)$, $K\{0\} = 0$

From properties of Kamal Transformation, we have

The Kamal Transformation of Caputo FD is $K[\,{}^c D_t^\alpha g(t)] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} g^k(0)$

And we have $\alpha \in (1,2]$ thus

$$K[\,{}^c D_a^\alpha u(x)] = v^{-\alpha} U(v) - \sum_{k=0}^{1} v^{k-\alpha+1} u^k(0) = v^{-\alpha} U(s) - v^{1-\alpha} u(0) - v^{2-\alpha} u'(0)$$

From Boundary condition $u(0) = 0$ so
$K[\,{}^c D_a^\alpha u(x)] = v^{-\alpha} U(s) - v^{1-2\alpha} u'(0)$

Then $K\{{}_0^C D_x^\alpha u(x)\} + K\{M(\lambda^2 - 1)u(x)\} = K\{0\}$

$\rightarrow v^{-\alpha} U(v) - v^{2-\alpha} u'(0) + M(\lambda^2 - 1)U(v) = 0$

$\rightarrow v^{-\alpha} U(v) + M(\lambda^2 - 1)U(s) = v^{2-\alpha} u'(0)$

$\rightarrow U(v)\left(v^{-\alpha} + M(\lambda^2 - 1)\right) = Av^{2-\alpha}$ Where $A = u'(0)$

$\rightarrow U(v) = \frac{Av^{2-\alpha}}{v^{-\alpha} + M(\lambda^2 - 1)} = \frac{Av^2}{(1 - M(1 - \lambda^2)v^\alpha)}$

The corresponding inverse Kamal transform is $K\big(g(t)\big) = G(v) \Leftrightarrow g(t) = K^{-1}(G(v))$

If it is exist we can take Inverse Kamal for both sides we obtain

$K^{-1}\{U(v)\} = K^{-1}\left\{\frac{Av^2}{(1-M(1-\lambda^2)v^\alpha)}\right\}$ Since $K\{u(x)\} = U(v) \rightarrow u(x) = K^{-1}\{U(v)\}$

So we get $u(x) = K^{-1}\left\{\frac{Av^2}{(1-M(1-\lambda^2)v^\alpha)}\right\}$, A *is constant* , $\alpha \in (1,2]$, λ *is complex number*

The Fractional Boundary Value Problem answer is now available. Between 1 and 2 is

$$u(x) = K^{-1}\left\{\frac{Av^2}{(1-M(1-\lambda^2)v^\alpha)}\right\} \quad ; x \in [0, a] \quad , \quad \alpha \in (1,2]$$

Such that $A = u'(0)$ is any non zero constant.

If $\alpha = 2 \rightarrow u(x) = K^{-1}\left\{\frac{Av^2}{(1-M(1-\lambda^2)v^2)}\right\} = \frac{A}{\sqrt{M(1-\lambda^2)}} sinh\left(\sqrt{M(1-\lambda^2)}x\right)$

Case 2: Variable Coefficients

In this case we assumed that the weight function $p(x) = 1$ and $q(x)$ any continuous function

And suppose that $q(x)u(x) = f(x)$

Now the Fractional differential equation become

$$-{}_0^C D_x^\alpha u(x) + f(x) = \lambda^2 u(x) \quad \rightarrow {}_0^C D_x^\alpha u(x) + \lambda^2 u(x) = f(x)$$

By applying Kamal transformation for above equation, we obtain

$$K\{{}_0^C D_x^\alpha u(x)\} + K\{\lambda^2 u(x)\} = K\{f(x)\}$$

Let $K\{u(x)\} = \int_0^\infty e^{\frac{-t}{v}} u(t)dt = U(v)$ and $K\{f(x)\} = \int_0^\infty e^{\frac{-t}{v}} f(t)dt = F(v)$

From properties of Kamal transformation for $\alpha \in (1,2]$ we have

$$K[\,^{c}D_{a}^{\alpha}u(x)] = v^{-\alpha}U(v) - \sum_{k=0}^{1} v^{k-\alpha+1}u^{k}(0) = v^{-\alpha}U(s) - v^{1-\alpha}u(0) - v^{2-\alpha}u'(0)$$

From Boundary condition $u(0) = 0$ so

$K[\,^{c}D_{a}^{\alpha}u(x)] = v^{-\alpha}U(s) - v^{1-2\alpha}u'(0)$ *Then* $K\{{}_{0}^{C}D_{x}^{\alpha}u(x)\} + K\{\lambda^{2}u(x)\} = K\{f(x)\}$

$$\rightarrow v^{-\alpha}U(v) - v^{2-\alpha}u'(0) + \lambda^{2}U(v) = F(v)$$

$$\rightarrow v^{-\alpha}U(v) + \lambda^{2}U(v) = F(v) + v^{2-\alpha}u'(0)$$

$\rightarrow U(v)(v^{-\alpha} + \lambda^{2}) = F(v) + Av^{2-\alpha}$ Where $A = u'(0)$

$$\rightarrow U(v) = \frac{Av^{2-\alpha}}{v^{-\alpha}+\lambda^{2}} + \frac{G(v)}{v^{-\alpha}+\lambda^{2}} = \frac{Av^{2}}{1+\lambda^{2}v^{\alpha}} + \frac{v^{\alpha}F(v)}{1+\lambda^{2}v^{\alpha}}$$

If it exists, we can use the Inverse Kamal transform to obtain

$K\{u(x)\} = U(v) \rightarrow u(x) = K^{-1}\{U(v)\}$ *and* $K\{f(x)\} = F(v) \rightarrow f(x) = K^{-1}\{F(v)\}$

$$K^{-1}\{U(v)\} = K^{-1}\left\{\frac{Av^{2}}{1+\lambda^{2}v^{\alpha}}\right\} + K^{-1}\left\{\frac{v^{\alpha}F(v)}{1+\lambda^{2}v^{\alpha}}\right\}.$$

For more information about Inverse Kamal Transform you can see [9], [25]

5. Illustrative examples

This section lists three problems that show how useful the Kamal transformation is. These examples demonstrate how to find precise solutions for linear fractional differential equations of fractional order in both the Riemann-Liouville and Liouville-Caputo senses. The examples mentioned were first addressed in references [13], [14] using Laplace transformation methods and other techniques. However, in this paper, we introduce a new method to solve these problems using the Kamal transformation method.

Example 5.1: Consider the fractional boundary value problem

$$-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t) + \frac{1}{10}u(t) = \lambda^{2}\frac{1}{10}u(t)\ ; \qquad 0 \le t \le 1$$

$u(0) = 0\,, \qquad u'(1) - i\lambda u(1) = 0$

Solution: we have $M = \frac{1}{10}$

Now $\quad -{}_0^CD_t^{\frac{3}{2}}u(t) + \frac{1}{10}u(t) = \frac{1}{10}\lambda^2 u(t) \rightarrow {}_0^CD_t^{\frac{3}{2}}u(t) = \frac{1}{10}(1-\lambda^2)u(t)$

Take Kamal for both sides we get

$K\left\{{}_0^CD_t^{\frac{3}{2}}u(t)\right\} = K\{\frac{1}{10}(1-\lambda^2)u(t)\}$ from properties of the Kamal Transformation,

The Kamal Transformation of Caputo FD is

$K[\,{}^cD_a^{\alpha}u(t)] = v^{-\alpha}U(v) - \sum_{k=0}^{1} v^{k-\alpha+1}u^k(0) = v^{-\alpha}U(v) - v^{1-\alpha}u(0) - v^{2-\alpha}u'(0),$

So $K\left\{{}_0^CD_t^{\frac{3}{2}}u(t)\right\} = v^{-\frac{3}{2}}U(v) - v^{-\frac{1}{2}}u(0) - v^{\frac{1}{2}}u'(0),$

$K\left\{{}_0^CD_t^{\frac{3}{2}}u(t)\right\} = \left\{v^{-\frac{3}{2}}U(v) - v^{\frac{1}{2}}u'(0)\right\},$

Now $v^{-\frac{3}{2}}U(v) - v^{\frac{1}{2}}u'(0) = \frac{1}{10}(1-\lambda^2)U(v),$

$\rightarrow \left(v^{-\frac{3}{2}} + \frac{1}{10}(\lambda^2 - 1)\right)U(v) = v^{\frac{1}{2}}u'(0)$, where $u'(0) \neq 0$

$U(v) = \frac{v^{\frac{1}{2}}u'(0)}{v^{-\frac{3}{2}} + \frac{1}{10}(\lambda^2-1)} \rightarrow U(v) = \frac{gv^2}{1+\frac{1}{10}(\lambda^2-1)v^{\frac{3}{2}}}$, where $g = u'(0)$,

Apply inverse Kamal transform for above equation we obtain

$u(t) = K^{-1}\big(U(v)\big) = K^{-1}\left(\frac{gv^2}{1+\frac{1}{10}(\lambda^2-1)v^{\frac{3}{2}}}\right)$

See [9], [25], For inverse Kamal and find the solution.

On the other hand, we can get some result easily by using Fractional integral operator before using Kamal transform, we will explain below

$$-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)+\frac{1}{10}u(t)=\lambda^{2}\frac{1}{10}u(t)$$

${}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)=\frac{1}{10}(1-\lambda^{2})u(t)$ apply integral of order $\frac{3}{2}$ for equation we obtain

$$I^{\frac{3}{2}}{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)=u(t)-\sum_{k=0}^{1}\frac{t^{k}}{n!}f^{k}(0)=u(t)-u(0)-tu'(0)=u(t)-tu'(0)$$

Now $u(t)-tu'(0)=\frac{1}{10}(1-\lambda^{2})I^{\frac{3}{2}}(u(t))$

Take Kamal Transformation for both sides $K\{u(t)\}-bK\{t\}=\ \frac{1}{10}(1-\lambda^{2})K\{I^{\frac{3}{2}}(u(t))\}$

From properties Kamal transform for fractional integral we have

$K[I^{\alpha}g(t)]=v^{\alpha}G(v)$ So $U(v)-bv^{2}=\frac{1}{10}(1-\lambda^{2})\,v^{\frac{3}{2}}U(v)$,

$U(v)=\frac{bv^{2}}{1+\frac{1}{10}(\lambda^{2}-1)v^{\frac{3}{2}}}$ Then $u(t)=K^{-1}(U(v))=K^{-1}\left(\frac{bv^{2}}{1+\frac{1}{10}(\lambda^{2}-1)v^{\frac{3}{2}}}\right)$.

Implies that, the exact solution is $u(t)=gtE_{\frac{3}{2},2}\left(\frac{1}{10}(1-\lambda^{2})t^{\frac{3}{2}}\right)$.

Example 5.2:

$$-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)+u(t)=\lambda^{2}u(t);\qquad 0\le t\le 1\ ,1<\alpha\le 2$$

$$u(0)=0\,,\qquad u'(1)-\ i\lambda u(1)=0.$$

Solution: if we have $\lambda=i$ Now $-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)+\ u(t)=-u(t)\rightarrow\ {}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)=2u(t)$

Since $1<\alpha\le 2$, we have: $I^{\frac{3}{2}}\left[D^{\frac{3}{2}}u(t]\right)=2I^{\frac{3}{2}}[u(t)]\rightarrow u(t)+at+b=2I^{\frac{3}{2}}[u(t)]$

Taking Kamal transform for both sides we get

$U(v) + av^2 + bv = 2v^{\frac{3}{2}}[U(v)]$ so $U(v)(2v^{\frac{3}{2}} - 1) = av^2 + bv$

So $u(t) = aK^{-1}\left(\frac{v^2}{2v^{\frac{3}{2}}-1}\right) + bK^{-1}\left(\frac{v}{2v^{\frac{3}{2}}-1}\right).$

From the boundary conditions we can find a and b easily.

And the solution is $u(t) = \frac{c_1}{2}\left(\frac{t}{\Gamma(2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right)$

Example 5.3:

$$-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t) + u(t) = \lambda^2 u(t); \qquad t \in [0,1], \alpha \in (1,2]$$

$$u(0) = 0, \qquad u'(1) - i\lambda u(1) = 0.$$

Solution: if we have $\lambda = 1$ Now $-{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t) + u(t) = u(t)$ from simplification we get

${}_{0}^{C}D_{t}^{\frac{3}{2}}u(t) = 0$ Since $1 < \alpha \leq 2$, take Kamal transform for both sides we get

$K\{{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)\} = K\ \{0\}$ implies that $K\left\{{}_{0}^{C}D_{t}^{\frac{3}{2}}u(t)\right\} = \left\{v^{-\frac{3}{2}}U(v) - v^{\frac{1}{2}}u'(0)\right\}$

Now $v^{-\frac{3}{2}}U(v) - v^{\frac{1}{2}}u'(0) = 0 \rightarrow U(v) = av^2$, where $a = u'(0)$

Take inverse Kamal transform we get $u(t) = at$.

Conclusions

This study looks into Kamal's transformation approach, which is a useful technique for resolving differential fractional equations (FDEs). Moreover, providing benefits including lowering equation complexity, streamlining their forms, and managing FDEs with constant coefficients. It offers an alternative to current FDE methods and has the potential to make a significant contribution to fractional computation, despite its limitations like any other method. This study's novel work, the successful application of the Kamal approach to the Regge fractional order problem, yields simple and encouraging findings. Lastly, the Kamal transformation technique is a workable approach for addressing FDEs, and if they get more complex, it may prove to be an even more helpful tool for controlling FDEs.

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تطبيق طريقة تحويل كمال لحل مسألة رجى من رتبة الكسرية

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المستخلص

الهدف من هذا العمل هو استخدام تحويل كمال، وهي تقنية تحويل جديدة، لحل مسألة الترتيب الكسري T. Regge. يبحث بشكل رئيسي في كيفية حل المعادلات التفاضلية الكسرية (FDEs) باستخدام منهجية التحويل الخاصة بكمال، لا سيما عندما يتعلق الأمر بالترتيب الكسرى للقيمة الحدودية T. Regge. بالإضافة إلى ذلك، تم تطوير صيغ تحويل كمال للمشتقات الكسرية والتكاملات الكسرية، وتمت مناقشة مزايا وعيوب الطريقة، وتم إعطاء العديد من الأمثلة لإظهار قابلية تطبيق الطريقة. تقدم منهجية تحويل كمال حلولاً مضمونة لـ FDEs، والتي لها تأثير كبير على مجال الحساب الكسري، كما هو موضح في الأمثلة التي استخدمناها. وبعض الأمثلة تثبت صحة الطريقة بشكل أوضح من خاتمة المقال.