Confidence Intervals In Three-Way Repeated Measurements Model

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ABSTRACT

This study examines the three-way repeated measurements model and looks at the model's ANOVA. The test statistic for the hypothesis corresponding to each model factor is defined, as is the distribution of a sum of squares statistics. The chi-square transformation of the sum of squares distribution was used to determine the confidence interval for the variance of components. The application of the repeated measurements model to an experiment is the study's practical component.

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1. Introduction

Repeated measurements model is a term used to describe data in which the response variable for each experimental unit is observed on multiple occasions and possibly under different experimental conditions (see, vonesh and chinchilli) [15]. The repeated measurements model can be used todetermine the effect of any factor on student performance Or to determine the most effective teaching methods compared to the results obtained over many years. Confidence intervals are a statistical measure used to determine the range of values that is likely to contain a certain value of a population based on a sample taken from it. If the confidence intervals include a null value, then there is no statistically significant difference between the factors in the repeated measures model. otherwise, there is a statistically significant difference between the factors. If there isn't a zero in the confidence intervals. This indicates that the repeated measurements model factors have a minor but statistically significant difference. Confidence intervals are bigger in populations and samples with more variability. Timm [14] studies 2 multivariate analysis of variance of repeated measurements. Jones and narathong [5] study the estimation of the variance and covariance components in linear models. Davidian and giltinan [1] study analysis of repeated measurement data. Klonecki, and et al [6] study strictly positive estimators for variance components. Klonecki and zontek [7] improved estimators for the simultaneous estimation of variance components. Griffiths, and et al [3] study a comparison of the univariate and multivariate multilevel models for repeated measures. Filipe [2] studies the components of a variance model for the analysis of repeated measurements. Komarek and lesaffre [8] study the generalized linear mixed model with a penalized gaussian mixture. Martin, and et al [12] studied the mixed model for the analysis of repeated-measurement multivariate count data. Kori, and al-mouel [9] studied the expected mean square rate estimation of the repeated measurements model. Kurisu and otsu [10] study the linearization of the nonparametric deconvolution estimators for the repeated measurements model. Liang and li [11] optimize repeated-measures analysis of variance. Gokhale, and et al [4] used repeated measurements to predict cardiovascular risk in patients. Riaz, and et al [13] studied a simple method to construct a circular repeated measurement design class. The objective of this paper is to determine the confidence interval, the test statistic for the hypothesis corresponding to each factor of the model considered, and the repeated measurements model applied to an experiment.

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2. Setting up the model

The following model is used to analyze data and defined under different conditions.

$$\psi_{ijk} = \mu + \tau_{i(j)} + \gamma_{j(k)} + \zeta_{i(k)} + \epsilon_{ijk}, \quad \ldots(1)$$

where ψ_{ijk} is the response variable, $\tau_{i(j)}, \gamma_{j(k)}$, and $\zeta_{i(k)}$ are the random effects for all $i = 1, \ldots, x$, $j = 1, \ldots, y$, and $k = 1, \ldots, z$ and ϵ_{ijk} is the random error for all $i = 1, \ldots, x$, $j = 1, \ldots, y$, and $k = 1, \ldots, z$.

By assuming that $\tau_{i(j)}$ i.i.d $\sim N(0, \sigma^2_\tau)$, $\gamma_{j(k)}$ i.i.d $\sim N\left(0, \sigma^2_\gamma\right)$, $\zeta_{i(k)}$ i.i.d $\sim N\left(0, \sigma^2_\zeta\right)$, and ϵ_{ijk} i.i.d $\sim N(0, \sigma^2_\epsilon)$.

Now, by starting with this identity to partition SS_{total}

$$\psi_{ijk} - \bar{\psi}_{...} = \left(\bar{\psi}_{ij.} - \bar{\psi}_{i..}\right) + \left(\bar{\psi}_{.jk} - \bar{\psi}_{.j.}\right) + \left(\bar{\psi}_{i.k} - \bar{\psi}_{..k}\right) + \left(\psi_{ijk} - \bar{\psi}_{ij.} - \bar{\psi}_{.jk} - \bar{\psi}_{i.k} + \bar{\psi}_{i..} + \bar{\psi}_{.j.} + \bar{\psi}_{..k} - \bar{\psi}_{...}\right)$$

it follows that $SS_\tau, SS_\gamma, SS_\zeta$, and SS_ϵ are defined as follows

$$SS_\tau = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z}\left(\bar{\psi}_{ij.} - \bar{\psi}_{i..}\right)^2, \quad SS_\gamma = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z}\left(\bar{\psi}_{.jk} - \bar{\psi}_{.j.}\right)^2, \quad \ldots(2)$$

$$SS_\zeta = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z}\left(\bar{\psi}_{i.k} - \bar{\psi}_{..k}\right)^2, \quad \ldots(3)$$

and $$SS_\epsilon = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z}\left(\psi_{ijk} - \bar{\psi}_{ij.} - \bar{\psi}_{.jk} - \bar{\psi}_{i.k} + \bar{\psi}_{i..} + \bar{\psi}_{.j.} + \bar{\psi}_{..k} - \bar{\psi}_{...}\right)^2. \quad \ldots.(4)$$

3. Mean squares expectation

Expected mean square (EMS) is a statistical concept that is commonly used in the analysis of variance (ANOVA) method. In simple terms, EMS is the expected value of certain statistics that arise in partitions of sums of squares in ANOVA. It is utilized to determine the correct analysis statistics and is an essential tool in experimental design. In ANOVA, EMS plays a vital role in determining the appropriate test statistics.

$$E(SS_\tau) = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\bar{\psi}_{ij.} - \bar{\psi}_{i..}\right)^2$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\left(\tau_{i(j)} - \bar{\tau}_{i(.)}\right)^2 + \left(\bar{\gamma}_{j(.)} - \bar{\gamma}_{.(.)}\right)^2 + \left(\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..}\right)^2\right)$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} \frac{(y-1)}{yz}\left(z\sigma^2_\tau + \sigma^2_\gamma + \sigma^2_\epsilon\right)$$

$$= x(y-1)\left(z\sigma^2_\tau + \sigma^2_\gamma + \sigma^2_\epsilon\right),$$

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$$E(MS_\tau) = E\left(\frac{SS_\tau}{x(y-1)}\right) = \left(z\sigma_\tau^2 + \sigma_\gamma^2 + \sigma_\epsilon^2\right),$$

$$E(SS_\gamma) = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\bar{\psi}_{.jk} - \bar{\psi}_{.j.}\right)^2$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\left(\bar{\gamma}_{j(k)} - \bar{\gamma}_{j(.)}\right)^2 + \left(\bar{\zeta}_{.(k)} - \bar{\zeta}_{.(.)}\right)^2 + \left(\bar{\epsilon}_{.jk} - \bar{\epsilon}_{.j.}\right)^2\right)$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} \frac{(z-1)}{xz}\left(x\sigma_\gamma^2 + \sigma_\zeta^2 + \sigma_\epsilon^2\right)$$

$$= y(z-1)\left(x\sigma_\gamma^2 + \sigma_\zeta^2 + \sigma_\epsilon^2\right),$$

$$E(MS_\gamma) = E\left(\frac{SS_\gamma}{y(z-1)}\right) = \left(x\sigma_\gamma^2 + \sigma_\zeta^2 + \sigma_\epsilon^2\right),$$

$$E(SS_\zeta) = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E(\bar{\psi}_{i.k} - \bar{\psi}_{..k})^2$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\left(\bar{\tau}_{i(.)} - \bar{\tau}_{.(.)}\right)^2 + \left(\bar{\zeta}_{i(k)} - \bar{\zeta}_{.(k)}\right)^2 + (\bar{\epsilon}_{i.k} - \bar{\epsilon}_{..k})^2\right)$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} \frac{(x-1)}{xy}\left(\sigma_\tau^2 + y\sigma_\zeta^2 + \sigma_\epsilon^2\right)$$

$$= z(x-1)\left(\sigma_\tau^2 + y\sigma_\zeta^2 + \sigma_\epsilon^2\right),$$

$$E(MS_\zeta) = E\left(\frac{SS_\zeta}{z(x-1)}\right) = \left(\sigma_\tau^2 + y\sigma_\zeta^2 + \sigma_\epsilon^2\right),$$

$$E(SS_\epsilon) = \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\psi_{ijk} - \bar{\psi}_{ij.} - \bar{\psi}_{.jk} - \bar{\psi}_{i.k} + \bar{\psi}_{i..} + \bar{\psi}_{.j.} + \bar{\psi}_{..k} - \bar{\psi}_{...}\right)^2$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\left(\epsilon_{ijk} - \bar{\epsilon}_{ij.} - \bar{\epsilon}_{.jk} - \bar{\epsilon}_{i.k} + \bar{\epsilon}_{i..} + \bar{\epsilon}_{.j.} + \bar{\epsilon}_{..k} - \bar{\epsilon}_{...}\right)^2$$

$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} E\Big(\epsilon_{ijk}^2 - 2\epsilon_{ijk}\bar{\epsilon}_{ij.} - 2\epsilon_{ijk}\bar{\epsilon}_{.jk} - 2\epsilon_{ijk}\bar{\epsilon}_{i.k} + 2\epsilon_{ijk}\bar{\epsilon}_{i..} + 2\epsilon_{ijk}\bar{\epsilon}_{.j.} + 2\epsilon_{ijk}\bar{\epsilon}_{..k}$$
$$- 2\epsilon_{ijk}\bar{\epsilon}_{...} + \epsilon_{ij.}^2 + 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{.jk} + 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{i.k} - 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{i..} - 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{.j.} - 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{..k} + 2\bar{\epsilon}_{ij.}\bar{\epsilon}_{...} + \epsilon_{.jk}^2$$
$$+ 2\bar{\epsilon}_{.jk}\bar{\epsilon}_{i.k} - 2\bar{\epsilon}_{.jk}\bar{\epsilon}_{i..} - 2\bar{\epsilon}_{.jk}\bar{\epsilon}_{.j.} - 2\bar{\epsilon}_{.jk}\bar{\epsilon}_{..k} + 2\bar{\epsilon}_{.jk}\bar{\epsilon}_{...} + \epsilon_{i.k}^2 - 2\bar{\epsilon}_{i.k}\bar{\epsilon}_{i..} - 2\bar{\epsilon}_{i.k}\bar{\epsilon}_{.j.}$$
$$- 2\bar{\epsilon}_{i.k}\bar{\epsilon}_{..k} + 2\bar{\epsilon}_{i.k}\bar{\epsilon}_{...} + \epsilon_{i}^2 + 2\bar{\epsilon}_{i..}\bar{\epsilon}_{.j.} + 2\bar{\epsilon}_{i..}\bar{\epsilon}_{..k} - 2\bar{\epsilon}_{i..}\bar{\epsilon}_{...} + \epsilon_{.j.}^2 + 2\bar{\epsilon}_{.j.}\bar{\epsilon}_{..k} - 2\bar{\epsilon}_{.j.}\bar{\epsilon}_{...}$$
$$+ \epsilon_{..k}^2 - 2\bar{\epsilon}_{..k}\bar{\epsilon}_{...} + \epsilon_{...}^2\Big)$$

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$$= \sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z}\frac{(x-1)(y-1)(z-1)}{xyz}\sigma_{\epsilon}^{2}$$

$$= (x-1)(y-1)(z-1)\sigma_{\epsilon}^{2},$$

$$E(MS_{\epsilon}) = E\left(\frac{SS_{\epsilon}}{(x-1)(y-1)(z-1)}\right) = \sigma_{\epsilon}^{2}.$$

4. Sum of square distribution

The distribution of each sum square is as follows

$$\frac{SS_{\tau}}{(z\sigma_{\tau}^{2}+\sigma_{\gamma}^{2}+\sigma_{\epsilon}^{2})}\sim\chi_{x(y-1)}^{2}, \qquad \ldots(5)$$

$$\frac{SS_{\gamma}}{(x\sigma_{\gamma}^{2}+\sigma_{\zeta}^{2}+\sigma_{\epsilon}^{2})}\sim\chi_{y(z-1)}^{2}, \qquad \ldots(6)$$

$$\frac{SS_{\zeta}}{(\sigma_{\tau}^{2}+y\sigma_{\zeta}^{2}+\sigma_{\epsilon}^{2})}\sim\chi_{z(x-1)}^{2}, \qquad \ldots(7)$$

and $\frac{SS_{\epsilon}}{\sigma_{\epsilon}^{2}}\sim\chi_{(x-1)(y-1)(z-1)}^{2}$. ….(8)

5. Analysis of variance

In this section, the ANOVA table for the model is presented as follows:

Table 1.: Analysis of variance table

Source of Variance	**D.F.**	**SS**	**MS**	**F-ratio**	**F table value**
Random τ	$x(y-1)$	SS_{τ}	MS_{τ}	$\frac{MS_{\tau}}{MS_{\epsilon}}$	F[x(y-1),(x-1)(y-1)(z-1);1-α]
Random γ	$y(z-1)$	SS_{γ}	MS_{γ}	$\frac{MS_{\gamma}}{MS_{\epsilon}}$	F[y(z-1),(x-1)(y-1)(z-1);1-α]
Random ζ	$z(x-1)$	SS_{ζ}	MS_{ζ}	$\frac{MS_{\zeta}}{MS_{\epsilon}}$	F[z(x-1),(x-1)(y-1)(z-1);1-α]
Error	(x-1)(y-1)(z-1)	SS_{ϵ}	MS_{ϵ}		

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For each factor, the null hypotheses H_0^{τ}, H_0^{γ}, and H_0^{ζ} are rejected by using F-ratio test statistics $\frac{MS_{\tau}}{MS_{\epsilon}}, \frac{MS_{\gamma}}{MS_{\epsilon}}$, and $\frac{MS_{\zeta}}{MS_{\epsilon}}$, which means that the corresponding factors Random τ, Random γ, and Random ζ have active effect. Where, the null and alternative hypothesis for each factor is defined as follows:

$$\ldots(9) H_0^{\tau}: all\ z\sigma_{\tau}^2 + \sigma_{\gamma}^2 = 0, and\ H_1^{\tau}: all\ z\sigma_{\tau}^2 + \sigma_{\gamma}^2 > 0,$$

$$\ldots(10) H_0^{\gamma}: all\ x\sigma_{\gamma}^2 + \sigma_{\zeta}^2 = 0, and\ H_1^{\gamma}: all\ x\sigma_{\gamma}^2 + \sigma_{\zeta}^2 > 0,$$

$$\ldots(11) H_0^{\zeta}: all\ \sigma_{\tau}^2 + y\sigma_{\zeta}^2 = 0, and\ H_1^{\zeta}: all\ \sigma_{\tau}^2 + y\sigma_{\zeta}^2 > 0,$$

6. Confidence interval for variance components

In this section, the confidence interval for the variance of components is identified as follows:

Theorem 6.1. The confidence interval for σ_{ϵ}^2 is $\left[\frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}\right]$.

Proof. Since $\frac{SS_{\epsilon}}{\sigma_{\epsilon}^2} \sim \chi^2_{(x-1)(y-1)(z-1)}$, it follows that

$$p\left[\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]} < \frac{SS_{\epsilon}}{\sigma_{\epsilon}^2} < \chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}\right] = 1-\alpha,$$

$$p\left[\frac{1}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}} < \frac{\sigma_{\epsilon}^2}{SS_{\epsilon}} < \frac{1}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

$$p\left[\frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}} < \sigma_{\epsilon}^2 < \frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

thus, the confidence interval for σ_{ϵ}^2 is $\left[\frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_{\epsilon}}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}\right]$.

Theorem 6.2. The confidence interval for $z\sigma_{\tau}^2 + \sigma_{\gamma}^2$ is

$$\left[\frac{SS_{\tau}-SS_{\epsilon}F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_{\tau}-SS_{\epsilon}F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right].$$

Proof. since $\frac{SS_{\tau}}{(z\sigma_{\tau}^2+\sigma_{\gamma}^2+\sigma_{\epsilon}^2)} \sim \chi^2_{x(y-1)}$, then

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$$p\left[\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]} < \frac{SS_\tau}{(z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2)} < \chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}\right] = 1-\alpha,$$

$$p\left[\frac{1}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} < \frac{(z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2)}{SS_\tau} < \frac{1}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

$$p\left[\frac{SS_\tau}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} < z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2 < \frac{SS_\tau}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

$$p\left[\frac{SS_\tau}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} - \sigma_\epsilon^2 < z\sigma_\tau^2+\sigma_\gamma^2 < \frac{SS_\tau}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}} - \sigma_\epsilon^2\right] = 1-\alpha,$$

$$p\left[\frac{SS_\tau-\sigma_\epsilon^2\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} < z\sigma_\tau^2+\sigma_\gamma^2 < \frac{SS_\tau-\sigma_\epsilon^2\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

$$p\left[\frac{SS_\tau-\frac{SS_\epsilon\,\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}}}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} < z\sigma_\tau^2+\sigma_\gamma^2 < \frac{SS_\tau-\frac{SS_\epsilon\,\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

$$p\left[\frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}} < z\sigma_\tau^2+\sigma_\gamma^2 < \frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right] = 1-\alpha,$$

thus, the confidence interval for $z\sigma_\tau^2+\sigma_\gamma^2$ is

$$\left[\frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right].$$

By using the same way, the confidence interval for $x\sigma_\gamma^2+\sigma_\zeta^2$ is

$$\left[\frac{SS_\gamma-SS_\epsilon F\left[y(z-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[y(z-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_\gamma-SS_\epsilon F\left[y(z-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[y(z-1),\frac{\alpha}{2}\right]}}\right].$$

And the confidence interval for $\sigma_\tau^2+y\sigma_\zeta^2$ is

$$\left[\frac{SS_\zeta-SS_\epsilon F\left[z(x-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[z(x-1),1-\frac{\alpha}{2}\right]}}, \frac{SS_\zeta-SS_\epsilon F\left[z(x-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[z(x-1),\frac{\alpha}{2}\right]}}\right].$$

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Theorem 6.3. The confidence interval for $\frac{z\sigma_\tau^2+\sigma_\gamma^2}{\sigma_\epsilon^2}$ is

$$\left[\frac{SS_\epsilon-SS_\tau F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{SS_\tau F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]},\frac{SS_\epsilon-SS_\tau F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{SS_\tau F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}\right].$$

Proof. since $\frac{SS_\tau}{(z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2)}\sim\chi^2_{x(y-1)}$, and $\frac{SS_\epsilon}{\sigma_\epsilon^2}\sim\chi^2_{(x-1)(y-1)(z-1)}$, then $\frac{\sigma_\epsilon^2}{(z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2)}\frac{SS_\tau}{SS_\epsilon}\sim F[\vartheta_\tau,\vartheta_\epsilon]$,

where $\vartheta_\tau=x(y-1),\vartheta_\epsilon=(x-1)(y-1)(z-1)$

$$p\left[F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]<\frac{\sigma_\epsilon^2}{(z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2)}\frac{SS_\tau}{SS_\epsilon}<F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]\right]=1-\alpha,$$

$$p\left[\frac{SS_\epsilon}{F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau}<\frac{z\sigma_\tau^2+\sigma_\gamma^2+\sigma_\epsilon^2}{\sigma_\epsilon^2}<\frac{SS_\epsilon}{F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}\right]=1-\alpha,$$

$$p\left[\frac{SS_\epsilon}{F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau}<\frac{z\sigma_\tau^2+\sigma_\gamma^2}{\sigma_\epsilon^2}+1<\frac{SS_\epsilon}{F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}\right]=1-\alpha,$$

$$p\left[\frac{SS_\epsilon-F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau}{F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau}<\frac{z\sigma_\tau^2+\sigma_\gamma^2}{\sigma_\epsilon^2}<\frac{SS_\epsilon-F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}{F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}\right]=1-\alpha,$$

thus, The confidence interval for $\frac{z\sigma_\tau^2+\sigma_\gamma^2}{\sigma_\epsilon^2}$ is $\left[\frac{SS_\epsilon-F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau}{F\left[\vartheta_\tau,\vartheta_\epsilon,1-\frac{\alpha}{2}\right]SS_\tau},\frac{SS_\epsilon-F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}{F\left[\vartheta_\tau,\vartheta_\epsilon,\frac{\alpha}{2}\right]SS_\tau}\right]$.

By using the same way, the confidence interval for $\frac{x\sigma_\gamma^2+\sigma_\zeta^2}{\sigma_\epsilon^2}$ is:

$$\left[\frac{SS_\epsilon-SS_\gamma F\left[y(z-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{SS_\gamma F\left[y(z-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]},\frac{SS_\epsilon-SS_\gamma F\left[y(z-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{SS_\gamma F\left[y(z-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}\right].$$

And the same way, the confidence interval for $\frac{\sigma_\tau^2+y\sigma_\zeta^2}{\sigma_\epsilon^2}$ is:

$$\left[\frac{SS_\epsilon-SS_\zeta F\left[z(x-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{SS_\zeta F\left[z(x-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]},\frac{SS_\epsilon-SS_\zeta F\left[z(x-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{SS_\zeta F\left[z(x-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}\right].$$

7. The experiment

The experiment included two levels of i, i_1 and i_2, also there are three levels of j, j_1 , j_2 and j_3, and three levels of k, k_1 , k_2 and k_3.

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Table2.: The experiment data

levels		k_1	k_2	k_3
i_1	j_1	81.6	87.8	91.2
	j_2	66.0	70.1	66.0
	j_3	77.2	90.0	85.3
i_2	j_1	68.9	76.3	77.1
	j_2	101.1	117.8	87.4
	j_3	83.3	62.5	92.1

$,\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\psi_{ijk}^{2} = 125224.45$

$+\psi_{12.}^{2}+\psi_{13.}^{2}+\psi_{21.}^{2}+\psi_{22.}^{2}+\psi_{23.}^{2}$ ...(12)$\sum_{i=1}^{2}\sum_{j=1}^{3}\psi_{ij.}^{2} = \psi_{11.}^{2}$

$$= 260.6^2 + 202.1^2 + 252.5^2 + 222.3^2 + 306.3^2 + 237.9^2$$

$$= 372346.41,$$

$+\psi_{1.2}^{2}+\psi_{1.3}^{2}+\psi_{2.1}^{2}+\psi_{2.2}^{2}+\psi_{2.3}^{2}$ ....(13)$\sum_{i=1}^{2}\sum_{k=1}^{3}\psi_{i.k}^{2} = \psi_{1.1}^{2}$

$$= 224.8^2 + 247.9^2 + 242.5^2 + 253.3^2 + 256.6^2 + 256.6^2$$

$$= 366643.71,$$

$+\psi_{.12}^{2}+\psi_{.13}^{2}+\psi_{.21}^{2}+\psi_{.22}^{2}+\psi_{.23}^{2} + \psi_{.31}^{2} + \psi_{.32}^{2} + \psi_{.33}^{2}$ ...(14)$\sum_{j=1}^{3}\sum_{k=1}^{3}\psi_{.jk}^{2} = \psi_{.11}^{2}$

$$= 150.5^2 + 164.1^2 + 168.3^2 + 167.1^2 + 187.9^2 + 153.4^2 + 160.5^2 + 152.5^2 + 177.4^2$$

$$= 245151.59,$$

$+\psi_{2..}^{2}$ ...(15)$\sum_{i=1}^{2}\psi_{i..}^{2} = \psi_{1..}^{2}$

$$= 715.2^2 + 766.5^2$$

$$= 1099033.29,$$

$+\psi_{.2.}^{2}+\psi_{.3.}^{2}$ ...(16)$\sum_{j=1}^{3}\psi_{.j.}^{2} = \psi_{.1.}^{2}$

$$= 482.9^2 + 508.4^2 + 490.4^2$$

$$= 732155.13,$$

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$+\psi^2_{..2}+\psi^2_{..3}$ ....(17)$\sum_{k=1}^{3}\psi^2_{..k} = \psi^2_{..1}$

$= 478.1^2 + 504.5^2 + 499.1^2$

$= 732200.67,$

....(18)$\psi^2_{...} = (\psi_{1..} + \psi_{2..})^2$

$= (715.2 + 766.5)^2$

. $= 2195434.89$

$SS_\tau = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\bar{\psi}_{ij.} - \bar{\psi}_{i..}\right)^2$

$= \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{3}\psi^2_{ij.} - \frac{1}{bn}\sum_{i=1}^{2}\psi^2_{i..}$ ....(19)

$= 2000.66,$

$SS_\gamma = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\bar{\psi}_{.jk} - \bar{\psi}_{.j.}\right)^2$

...(20) $= \frac{1}{a}\sum_{j=1}^{3}\sum_{k=1}^{3}\psi^2_{.jk} - \frac{1}{an}\sum_{j=1}^{3}\psi^2_{.j.}$

$= 549.94,$

$SS_\zeta = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}(\bar{\psi}_{i.k} - \bar{\psi}_{..k})^2$

...(21) $= \frac{1}{b}\sum_{i=1}^{2}\sum_{k=1}^{3}\psi^2_{i.k} - \frac{1}{ab}\sum_{k=1}^{3}\psi^2_{..k}$

$= 181.125,$

$SS_\epsilon = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\psi_{ijk} - \bar{\psi}_{ij.} - \bar{\psi}_{.jk} - \bar{\psi}_{i.k} + \bar{\psi}_{i..} + \bar{\psi}_{.j.} + \bar{\psi}_{..k} - \bar{\psi}_{...}\right)^2$

$= \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\psi^2_{ijk} - \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{3}\psi^2_{ij.} - \frac{1}{a}\sum_{j=1}^{3}\sum_{k=1}^{3}\psi^2_{.jk} - \frac{1}{b}\sum_{i=1}^{2}\sum_{k=1}^{3}\psi^2_{i.k} +$

...(22)$\frac{1}{bn}\sum_{i=1}^{2}\psi^2_{i..} + \frac{1}{an}\sum_{j=1}^{3}\psi^2_{.j.} + \frac{1}{ab}\sum_{k=1}^{3}\psi^2_{..k} - \frac{1}{abn}\psi^2_{...}$

.= *524.12*

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Table 3.: ANOVA data

Source of Variance	D.F.	SS	MS	F-ratio	F table value
Random τ	4	2000.66	500.165	3.81718	6.38823
Random γ	6	549.94	91.656	0.6995	6.16313
Random ζ	3	181.125	60.375	0.46077	6.59138
Error	4	524.12	131.03		

The confidence interval for σ_ϵ^2 is [47.0345,1081.9559].

The least significant difference (LSD) for each factor is defined as follows:

$$....(23)\ LSD_{0.05}(\tau) = \sqrt{\frac{2\ MS_\epsilon}{z}} \times t_{0.025,(x-1)(y-1)(z-1)}$$

$= 9.346 \times 2.776$

$= 25.94,$

$$....(24)\ LSD_{0.05}(\gamma) = \sqrt{\frac{2\ MS_\epsilon}{x}} \times t_{0.025,(x-1)(y-1)(z-1)}$$

$= 11.45 \times 2.776$

$= 31.7852,$

$$....(25)\ LSD_{0.05}(\zeta) = \sqrt{\frac{2\ MS_\epsilon}{y}} \times t_{0.025,(x-1)(y-1)(z-1)}$$

$= 9.346 \times 2.776$

$= 25.94.$

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Table 4.: Mean difference among the levels $\psi_{ij.}$

	$\psi_{11.}$	$\psi_{13.}$	$\psi_{23.}$	$\psi_{21.}$	$\psi_{12.}$
$\psi_{22.}$	45.7	53.8	68.4	84	104.2
$\psi_{11.}$		8.1	22.7	83.3	58.5
$\psi_{13.}$			14.6	30.2	50.4
$\psi_{23.}$				15.6	35.8
$\psi_{21.}$					20.2
$\psi_{12.}$					

Table 5.: Mean difference among the levels $\psi_{.jk}$

	$\psi_{.33}$	$\psi_{.13}$	$\psi_{.21}$	$\psi_{.12}$	$\psi_{.31}$	$\psi_{.23}$	$\psi_{.32}$	$\psi_{.11}$
$\psi_{.22}$	10.5	19.6	20.8	23.8	27.4	34.5	35.4	37.4
$\psi_{.33}$		9.1	10.3	13.4	16.9	24	24.9	27.4
$\psi_{.13}$			1.2	4.2	7.8	14.9	15.8	17.8
$\psi_{.21}$				3	6.6	13.7	14.6	16.6
$\psi_{.12}$					3.6	10.7	11.6	13.6
$\psi_{.31}$						7.1	8	10
$\psi_{.23}$							0.9	2.9
$\psi_{.32}$								2
$\psi_{.11}$								

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Table 6.: Mean difference among the levels $\psi_{i.k}$

	$\psi_{2.1}$	$\psi_{1.2}$	$\psi_{1.3}$	$\psi_{1.1}$
and $\psi_{2.3}\psi_{2.2}$	3.3	8.7	14.1	31.8
$\psi_{2.1}$		5.4	10.8	28.5
$\psi_{1.2}$			5.4	23.1
$\psi_{1.3}$				17.7
$\psi_{1.1}$				

By comparing the mean difference among the levels $\psi_{ij.}$, $\psi_{.jk}$, and $\psi_{i.k}$ with the corresponding LSD, it follows that the level i_2 along with j_2 produces significantly higher than any other level.

Conclusions

1. For the experiment, the confidence interval for σ_ϵ^2 is $[47.0345,1081.9559]$, $\varphi_{22.}$ produces significantly higher than any other level.

2. The confidence interval for σ_ϵ^2 is $\left[\frac{SS_\epsilon}{\chi^2_{\left[(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}},\frac{SS_\epsilon}{\chi^2_{\left[(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}}\right]$.

3. The confidence interval for $z\sigma_\tau^2+\sigma_\gamma^2$ is

$$\left[\frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),1-\frac{\alpha}{2}\right]}},\frac{SS_\tau-SS_\epsilon F\left[x(y-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[x(y-1),\frac{\alpha}{2}\right]}}\right].$$

4. The confidence interval for $x\sigma_\gamma^2+\sigma_\zeta^2$ is

$$\left[\frac{SS_\gamma-SS_\epsilon F\left[y(z-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[y(z-1),1-\frac{\alpha}{2}\right]}},\frac{SS_\gamma-SS_\epsilon F\left[y(z-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[y(z-1),\frac{\alpha}{2}\right]}}\right].$$

5. The confidence interval for $\sigma_\tau^2+y\sigma_\zeta^2$ is

$$\left[\frac{SS_\zeta-SS_\epsilon F\left[z(x-1),(x-1)(y-1)(z-1),1-\frac{\alpha}{2}\right]}{\chi^2_{\left[z(x-1),1-\frac{\alpha}{2}\right]}},\frac{SS_\zeta-SS_\epsilon F\left[z(x-1),(x-1)(y-1)(z-1),\frac{\alpha}{2}\right]}{\chi^2_{\left[z(x-1),\frac{\alpha}{2}\right]}}\right].$$

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فترات الثقة في نموذج القياسات المتكررة ثلاثي الاتجاه

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المستخلص

في هذه الورقة ، تمت دراسة نموذج القياسات المتكررة ثلاثي الاتجاه ، وتمت دراسة تحليل التباين للنموذج. تم تعريف توزيع إحصائيات مجموع المربعات، و تم تعريف إحصائية اختبار للفرضية المقابلة لكل عامل في النموذج. تم تحديد فترة الثقة لمركبات التباين باستخدام تحويل مربع كاي لتوزيع مجموع المربعات. الجانب العملي لهذه الدراسة هو تطبيق نموذج القياسات المتكررة على تجربة.

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