# Classification of degree three arcs in PG(2,19) up to stabilizer groups

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**ABSTRACT**

An $(n,3)$-arc $\mathcal{K}$ in projective plane $PG(2,q)$ of size $n$ and degree three is a set of n points such that every line in the plane meet it in less than or equal three points, also the arc $\mathcal{K}$ is complete if it is not contained in $(n+1,3)$-arc. In this paper, the classification of degree three arcs in $PG(2,19)$ is introduced in details according to their stabilizer groups. The motivation for working in the projective plane of order 19 is twofold. First, the size of the largest $(n,3)$-arc is not known. Second, the number of $(n,3)$-arcs is significantly higher in the projective plane of order 19 than it is in the projective plane of order $q$ for $q<19$, giving a large number of $(n,3)$-arcs for the study.

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## 1. Introduction

Let $\mathbb{F}_q$ be the Galois field with $q$ elements, and $V(3,q)$ be the 3-dimensional vector space over the field $\mathbb{F}_q$. The corresponding projective plane of $V(3,q)$ is denoted by $\mathbb{P}_q^2$. The points $(x_1, x_2, x_3)$ of $\mathbb{P}_q^2$ are the 1-dimensional subspaces of $V(3,q)$. Subspaces of dimension two of $V(3,q)$ are called lines which has the form $\mathbb{V}(ax_1 + bx_2 + cx_3)$. The number of points and the number of lines in $\mathbb{P}_q^2$ is $q^2 + q + 1$. There are $q + 1$ points on every line and $q + 1$ lines through every point. Many researches have been studies the subject of projective geometries for examples see [1], the tools of this paper Gap-Groups, Algorithms, programming a system for computational discrete algebra [17].

A full classification of $(n, 3)$-arcs are given by Cook [2]. For $q = 11$, a maximal $(n, 3)$-arcs has been found by Marcugini [3]. For $q = 13$, the maximal arc has been found in [4]. Many studies and improvements have been given to get a largest $(n, r)$-arc of size two, three, four, etc. For more detailed to the size of $(n,3)$-arcs see [5]. A full classification of the complete k-arcs of PG (2, 23) and PG (2, 25) is done by Coolsaet and Sticker [6]. Also, the complete (k, 3)-arcs of $PG\ (2, q)$, $q \leq\ 13$ is given by Coolsaet and Sticker [7]. The classification of the $(k, 3)$-arcs in $PG(2,37)$ is presented [8]. In [9], the construction of $(4v, 3)$-arcs that produced from irreducible plane-cubic is discussed for all values of $q$, where $7 \leq q \leq 37$. Many large complete arcs such as (46,3)-arcs, (67,4)-arcs, (91,5)-arcs, (201,8)-arcs, (226,9)-arcs, (469,16)-arcs and (488,17)-arcs in $PG\ (2, 37)$ are obtained in [10]. Furthermore, large size for the complete $(k, 3)$-arcs in the projective plane of order nineteen $PG(2,19)$ using the method of secants distributions [11].

The main aim of this paper is to determine the stabilizer group of each an equivalent $(n, 3)$-arcs in $\mathbb{P}_9^2$, where $n > 4$. Classify $(n, 3)$-arcs in way, give us ($(n, 3)$-arcs of large size in $\mathbb{P}_{19}^2$. The classification technique is based on constructed the special arcs in $\mathbb{P}_q^2$ to find the largest size of complete $(n, 3)$-arcs [12]. In fact, up to stabilizer group we obtained that the largest size of complete $(n, 3)$-arc is equal to 30.

## 2. Preliminaries

### 2.2 Group Theory

A *group* is a set $\boldsymbol{G}$ together with an operation $*$ satisfying the following requirements:

1. for each pair $(\boldsymbol{x}, \boldsymbol{y})$ of elements of $\boldsymbol{G} \times \boldsymbol{G}$, $\boldsymbol{x} * \boldsymbol{y}$ is an element of $\boldsymbol{G}$;

2. for all elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of $\boldsymbol{G}$, $(\boldsymbol{x} * \boldsymbol{y}) * \boldsymbol{z} = \boldsymbol{x} * (\boldsymbol{y} * \boldsymbol{z})$;
3. there is an element $\boldsymbol{e} \in \boldsymbol{G}$ such that $\boldsymbol{e} * \boldsymbol{g} = \boldsymbol{g} = \boldsymbol{g} * \boldsymbol{e}$ for all $\boldsymbol{g} \in \boldsymbol{G}$;
4. given an element $\boldsymbol{g} \in \boldsymbol{G}$, there is an element $\boldsymbol{g}' \in \boldsymbol{G}$ such that $\boldsymbol{g}' * \boldsymbol{g} = \boldsymbol{e} = \boldsymbol{g} * \boldsymbol{g}'$.

A group $G$ is *abelian* if, for all $g, g' \in G$, $g' * g = g * g'$. A *cyclic group* is a group generated by a single element; that is, a group consisting of all powers of one of its elements. If $G$ is a cyclic group generated by $g \in G$, we write $G = \langle g \rangle$. A finite group $G$ of order $n$ is cyclic if and only if it contains an element of order $n$. Also, a group of prime order is cyclic [13].

**Definition ([13])** An *action* of a group $(G,*)$ on a set $K$ is a function $\varphi : K \times G \to K$ with the following properties:

1. $((x, g)\varphi, h)\varphi = (x, g * h)\varphi$ for all $x \in K$ and $g, h \in G$;
2. $(x, e)\varphi = x$ for all $x \in K$, where $e$ is the identity of $G$;
3. $((x, g)\varphi, g')\mu = ((x, g')\varphi, g)\varphi = x$ for all $x \in K; g \in G$, where $g'$ is the inverse of the element $g$ in $G$.

**Definition ([13])** Let $G, H$ be two groups. A *homomorphism* $h : G \to H$ is a function $h$ from $G$ to $H$ that satisfies the condition, for all $g_1, g_2 \in G$,

$$(g_1 g_2)h = (g_1 h)(g_2 h) \qquad \mathbf{(1)}$$

A homomorphism that is one-to-one and onto is called an *isomorphism*. In this case, $G$ and $H$ is called *isomorphic* groups. A bijective homomorphism $h$ from a group to itself is called an *automorphism*.

**Definition ([14])** A bijection $h : X \to X$ is a permutation on $X$. The set of all permutations on $X$ is denoted by $S(X)$.

The set $S(X)$ forms a group under the usual composition of functions. If $X = \{1,2, \dots, n\}$ or any set with cardinality $n$, then $S(X)$ is written as $\mathfrak{S}_n$ which is called the symmetric group on $n$ symbols.

### 2.2 Projective Spaces Over a Finite Field

A field is a set $\mathbb{F}$ closed under two operations $+, \times$ such that

1. $(\mathbb{F}, +)$ is an abelian group with identity $\mathbf{0}$;

2. $(\mathbb{F}\backslash\{0\},\times)$ is an abelian group with identity 1;
3. $a(b+c) = ab + ac$; $(a+b)c = ac + bc$, for all $a,b,c \in \mathbb{F}$.

Let $\mathbb{F}_q$ denotes a field of $q$ elements. Let $V = V(n+1,q)$ be an $(n+1)$-dimensional vector space over a field $\mathbb{F}_q$ with zero element $\mathbf{0}$. Consider the equivalence relation on the elements of $V^* = V\backslash\{\mathbf{0}\}$ whose equivalence classes are the 1-dimensional subspaces of $V$ with zero removed. In fact, if $v_1, v_2 \in V^*$ where $v_1 = (x_1, \dots, x_{n+1})$ and $v_2 = (y_1, \dots, y_{n+1})$, then $v_1$ is equivalent to $v_2$ if $v_2 = \alpha v_1$ for some $\alpha \in \mathbb{F}_q\backslash\{0\}$; that is, $y_i = \alpha x_i$ for all $i$. The set of all equivalence classes is the *$n$-dimensional projective space* over $\mathbb{F}_q$ and is denoted by $PG(n,q)$.

The elements of $PG(n,q)$ are called *points*; the equivalence class of the vector $v$ is the point $P(v)$. In this case, we say that $v$ is a *coordinate vector* for $P(v)$ or that $v$ is a vector representing $P(v)$. By definition, $P(\alpha v) = P(v)$ for all $\alpha \in \mathbb{F}_q\backslash\{0\}$.

**Definition ([15])** A collineation of projective plane $\Pi = PG(2,q)$ is a map $\varphi$ of $\Pi$ onto $\Pi$ such that $\varphi$ is bijection;

1. $\varphi$ maps points onto points and lines onto lines;
2. if $P$ and $\ell$ are an incident point and line in $\Pi$, then $\varphi(P)$ and $\varphi(\ell)$ are incident.

**Theorem ([15])** If $\{P_0, P_1, \dots, P_{n+1}\}$, $\{P_0', P_1', \dots, P_{n+1}'\}$ are two sets of $n+2$ points of $PG(n,q)$ such that no $n+1$ points chosen from the same set lie in a subspace of dimension $n-1$, then there exists a unique projectivity $\mathfrak{T}$ such that $P_i' = P_i\mathfrak{T}$, for all $i = 0,1,\dots,n+1$.

The Theorem above is called the Fundamental Theorem of Projective Geometry.

**Definition ([15])** For any positive integer $n$, the *general linear group*, $GL(n,q)$, is the set of all invertible $n \times n$ matrices over $\mathbb{F}_q$ under matrix multiplication. The order of the group $GL(n,q)$ is $(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$.

The *projective general linear group* $PGL(n,q)$ is the group of projectivities of $PG(n-1,q)$ with respect to the operation of composition of maps.

**Definition ([15])** The *special linear group* $SL(n,q)$ is the subgroup of the group $GL(n,q)$ consisting of all non-singular matrices. The *projective special linear group* $PSL(n,q)$ is the quotient group $SL(n,q)/Z$, where $Z$ is the subgroup of scalar matrices in $SL(n,q)$.

### 2.3 Projective Planes

Consider the projective plane $PG(2,q)$ over the field $\mathbb{F}_q$. The projective plane $PG(2,q)$ contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines. There are exactly $q + 1$ points on each line, and $q + 1$ lines through each point. The points and lines of $PG(2,q)$ satisfy the following axioms of a projective plane:

1. every two distinct points are on a unique common line;
2. every two distinct lines contain a unique common point;
3. there are four distinct points, no three of which are on a common line.

A point $P(x,y,z)$ is incident with a line $\ell(k,l,m)$ if and only if $kx + ly + mz = 0$.

There is a non-singular matrix $\mathfrak{T} = (\alpha_{ij})$ where $\alpha_{ij} \in \mathbb{F}_q$ associated to each bijection from $PG(2,q)$ to $PG(2,q)$ that maps the point $P(v) = P(x,y,z)$ to the point $P(v') = P(x',y',z')$ and the line $\ell(U) = \ell(k,l,m)$ to the line $\ell(U'^t) = \ell(k',l',m')^t$ with $v' = v\mathfrak{T}$ and $U'^t = \mathfrak{T}^{-1}U^t$. In other words,

$$(x',y',z') = (x,y,z)\begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix} \qquad (2)$$

and

$$\begin{bmatrix}k'\\ l'\\ m'\end{bmatrix} = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix}^{-1}\begin{bmatrix}k\\ l\\ m\end{bmatrix} \qquad (3)$$

Any such a bijection is called a *projectivity* or *projective linear transformation*. Further, a projectivity preserves the incidence between points and lines.

The group of all projectivities of $PG(2,q)$ is the projective general linear group $PGL(3,q)$ and has order $(q^3 - 1)(q^2 - 1)q^3$. From the Fundamental Theorem of Projective Geometry, a projectivity is uniquely determined by the four images of the vertices of a quadrangle.

**Definition ([13])** A $(k,n)$-arc $\mathcal{K}$ in a projective plane $PG(2,q)$ is a set of $k$ points such that some line of the plane meets $\mathcal{K}$ in $n$ points but such that no line meets $\mathcal{K}$ in more than $n$ points, where $n \geq 2$.

Throughout, $\mathbb{P}_q^2$ will denote the projective plane $PG(2, q)$. A line $\ell$ of $\mathbb{P}_q^2$ is an $i$-secant of a $(k, n)$-arc $\mathcal{K}$ if $\ell$ intersects $\mathcal{K}$ in $i$ points. Let $\tau_i$ be the total number of $i$-secants to $\mathcal{K}$. The number of $i$-secants to $\mathcal{K}$ through a point $P$ of $\mathcal{K}$ is denoted by $\rho_i$ or $\rho_i(P)$. Moreover, $\sigma_i$ or $\sigma_i(Q)$ denotes the number of $i$-secants to $\mathcal{K}$ through a point $Q$ of $\mathbb{P}_q^2 \backslash \mathcal{K}$. A $(k, n)$-arc is complete if there is no $(k + 1, n)$-arc containing it. For more information about complete and incomplete $(k, n)$-arcs, one can see [16].

**Lemma ([15])** *For a $(k, n)$-arc $\mathcal{K}$, the following equations hold:*

$$\sum_{i=0}^{n} \tau_i = q^2 + q + 1; \qquad (4)$$

$$\sum_{i=1}^{n} i\tau_i = k(q + 1); \qquad (5)$$

$$\sum_{i=2}^{n} i(i-1)\tau_i = k(k-1); \qquad (6)$$

**Definition ([10])** The points out of a $(k, n)$-arc $\mathcal{K}$ in $\mathbb{P}_q^2$ which passes through it i-secant of $\mathcal{K}$ is called a point of index i**.**

## 1. Main Results

### **3.1** The classification of $(\boldsymbol{\nu}, \mathbf{3})$-arcs; $\boldsymbol{\nu} = \mathbf{5}, \mathbf{6}, \mathbf{7}$

First of all, let us give the following definitions which help us in our study.

**Definition ([9]).** Two $(k, n)$-arcs $\mathcal{K}_1$ and $\mathcal{K}_2$ in $\mathbb{P}_q^2$ are said to be stabilizer inequivalent if they have different stabilizer groups, that is, $\mathrm{Stab}(\mathcal{K}_1) \ncong \mathrm{Stab}(\mathcal{K}_2)$ where

$$Stab(\mathcal{K}) = \{\mathfrak{T} \in PGL(3, q) : \mathfrak{T}(\mathcal{K}) = \mathcal{K}\},$$

*for any $(k, n)$-arc $\mathcal{K}$ in $\mathbb{P}_{\boldsymbol{q}}^{\mathbf{2}}$.*

**Definition.** A point $P$ in $\mathbb{P}_{\boldsymbol{q}}^{\mathbf{2}}$ is called preferred point of type 3 if when it added to $\nu$-arc gives an arc of degree 3 in $\mathbb{P}_{\boldsymbol{q}}^{\mathbf{2}}$.

Let $\mathcal{R} = \{1,2,3,4\}$ be the 2-arc consisting of the indices of the points $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$ and $P_4 = (1,1,1)$. Henceforth, $\mathcal{D}$ denotes a set of preferred points of type 3 in $\mathbb{P}_{\boldsymbol{q}}^{\mathbf{2}}$ which can be add to $\mathcal{R} = \{1,2,3,4\}$ to produce arc of degree three.

In this subsection, the classification of $(\nu, 3)$-arcs, where $\nu = 5,6,7$, is established by classifying all such arcs up to stabilizer groups.

Let $\mathcal{R} = \{1,2,3,4\}$ be the 2-arc consisting of the indices of the points $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$ and $P_4 = (1,1,1)$. There is only one inequivalent (4,3)-arc in $\mathbb{P}_{19}^2$ up to stabilizer group, while there are 4 types of (5,3)-arcs during implementation our program. The 4 types are shown in Table 1.

**Table 1. Types of $(5,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_5^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_5^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_5^j$)** |
|---|---|---|
| $\mathcal{A}_5^1$ | 5 | $\mathbb{Z}_2$ |
| $\mathcal{A}_5^2$ | 35 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_5^3$ | 25 | $\mathbb{Z}_6$ |
| $\mathcal{A}_5^4$ | 150 | $\mathbf{D}_4$ |

Note that, $\text{Stab}(\mathcal{A}_5^1) = \langle M_5^1 \rangle$ and $\text{Stab}(\mathcal{A}_5^3) = \langle M_5^3 \rangle$, where

$$M_5^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_5^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The stabilizer group $\mathcal{A}_5^2$ consists of the following 4 projective matrices each one of them has order 2, say

$$M_{5,1}^2 = I,\ M_{5,2}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ M_{5,3}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } M_{5,4}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Moreover, the group Stab($\mathcal{A}_5^3$) is cyclic of order 6 and Stab($\mathcal{A}_5^3$) $= \langle M_5^3 \rangle$, where

$$M_5^3 = \begin{bmatrix} 0 & -1 & 1 \\ 7 & -8 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stabilizer group of $\mathcal{A}_5^4$ consists of the following 8 projective matrices, say

$$M_{5,1}^4 = I,\ M_{5,2}^4 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix},\ M_{5,3}^4 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix},\ M_{5,4}^4 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_{5,5}^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ M_{5,6}^4 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ M_{5,7}^4 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } M_{5,8}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

The order of $M_{5,1}^4$ is 1, the order of both $M_{5,2}^4$, $M_{5,4}^4$ is 4, and the order of each matrix in the remaining projective matrices is 2. So, Stab($\mathcal{A}_5^3$) isomorphic to $\mathbf{D}_4$.

Now, by adding 2 preferred points of type 3 to $\mathcal{R}$, we get (6,3)-arcs. In fact, there are 8 types of (6,3)-arcs up to stabilizer group, as shown in Table 2.

**Table 2. Types of $(6,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_6^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_6^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_6^j$)** |
|---|---|---|
| $\mathcal{A}_6^1$ | 5,6 | **I** |
| $\mathcal{A}_6^2$ | 5,12 | $\mathbb{Z}_2$ |
| $\mathcal{A}_6^3$ | 5,164 | $\mathbb{Z}_3$ |
| $\mathcal{A}_6^4$ | 35,56 | $\mathfrak{S}_3$ |
| $\mathcal{A}_6^5$ | 35,89 | $\mathbf{D}_4$ |

| | | |
|---|---|---|
| $\mathcal{A}_6^6$ | 25,133 | $\mathfrak{S}_3 \times \mathbb{Z}_3$ |
| $\mathcal{A}_6^7$ | 150,236 | $\mathfrak{S}_4$ |
| $\mathcal{A}_6^8$ | 40,89 | $\mathbb{Z}_6$ |

In Table 2, $\text{Stab}(\mathcal{A}_6^2) = \langle M_6^2 \rangle$, $\text{Stab}(\mathcal{A}_6^3) = \langle M_6^3 \rangle$ and $\text{Stab}(\mathcal{A}_6^8) = \langle M_6^8 \rangle$, where

$$M_6^2 = \begin{bmatrix} -4 & 4 & 0 \\ -3 & 4 & 0 \\ -6 & 4 & 2 \end{bmatrix}, M_6^3 = \begin{bmatrix} -5 & 6 & 0 \\ -5 & 5 & -8 \\ -5 & -3 & 0 \end{bmatrix} \text{ and } M_6^8 = \begin{bmatrix} 7 & 8 & 5 \\ 0 & 8 & 0 \\ 0 & 8 & -7 \end{bmatrix}.$$

The group $\text{Stab}(\mathcal{A}_6^4)$ is non-abelian of order 6. The following are all the projective matrices in $\text{Stab}(\mathcal{A}_6^4)$**:**

$$M_{6,1}^4 = I,\ M_{6,2}^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, M_{6,3}^4 = \begin{bmatrix} -5 & -5 & -9 \\ -4 & 5 & 0 \\ 5 & -5 & 0 \end{bmatrix}, M_{6,4}^4 = \begin{bmatrix} 1 & -9 & 9 \\ 0 & 9 & -8 \\ 0 & -9 & -9 \end{bmatrix},$$

$$M_{6,5}^4 = \begin{bmatrix} -5 & 5 & 0 \\ -4 & 5 & 0 \\ 5 & 5 & 9 \end{bmatrix} \text{ and } M_{6,6}^4 = \begin{bmatrix} 0 & -9 & -9 \\ 0 & -9 & 8 \\ 1 & -9 & 9 \end{bmatrix}.$$

Moreover, $\text{Stab}(\mathcal{A}_6^5) = \langle R, S \colon R^4 = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The group $\text{Stab}(\mathcal{A}_6^6)$ has 18 projective matrices, say

$$M_{6,1}^6 = I,\ M_{6,2}^6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}, M_{6,3}^6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 7 & -7 & 0 \end{bmatrix}, M_{6,4}^6 = \begin{bmatrix} -8 & 0 & 8 \\ 0 & -8 & 8 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_{6,5}^{6}=\begin{bmatrix}8&0&-7\\0&8&-7\\0&0&-7\end{bmatrix}, M_{6,6}^{6}=\begin{bmatrix}-7&8&0\\-7&8&-1\\-7&8&0\end{bmatrix}, M_{6,7}^{6}=\begin{bmatrix}-7&8&0\\1&0&0\\0&0&1\end{bmatrix}, M_{6,8}^{6}=\begin{bmatrix}8&0&-7\\8&0&0\\8&-8&0\end{bmatrix},$$

$$M_{6,9}^{6}=\begin{bmatrix}-1&-7&8\\-8&0&8\\0&0&1\end{bmatrix}, M_{6,10}^{6}=\begin{bmatrix}0&8&-7\\0&8&0\\-1&1&0\end{bmatrix}, M_{6,11}^{6}=\begin{bmatrix}1&7&-7\\1&7&0\\-7&7&0\end{bmatrix},$$

$$M_{6,12}^{6}=\begin{bmatrix}1&7&-7\\8&0&-7\\0&0&-7\end{bmatrix}, M_{6,13}^{6}=\begin{bmatrix}0&1&0\\-7&8&0\\0&0&1\end{bmatrix}, M_{6,14}^{6}=\begin{bmatrix}8&0&-8\\8&0&-7\\8&-8&0\end{bmatrix},$$

$$M_{6,15}^{6}=\begin{bmatrix}0&-8&8\\-1&-7&8\\0&0&1\end{bmatrix}, M_{6,16}^{6}=\begin{bmatrix}0&8&-8\\0&8&-7\\-1&1&0\end{bmatrix}, M_{6,17}^{6}=\begin{bmatrix}1&7&-8\\1&7&-7\\-7&7&0\end{bmatrix} \text{ and}$$

$$M_{6,18}^{6}=\begin{bmatrix}0&8&-7\\1&7&-7\\0&0&-7\end{bmatrix}.$$

The group $\mathrm{Stab}(\mathcal{A}_6^7)$ is non-abelian, and has 24 projective matrices. In fact, $\mathrm{Stab}(\mathcal{A}_6^7)=\langle R,S\rangle$, where

$$R=\begin{bmatrix}0&-1&1\\0&0&1\\-1&0&1\end{bmatrix} \text{ and } S=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

Adding 3 preferred points of type 3 to $\mathcal{R}$, give us (7,3)-arcs. Further, there are 9 types of (7,3)-arcs up to stabilizer group, as shown in Table 3.

**Table 3. Types of $(7,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_7^j=\mathcal{R}\cup\mathcal{D}$**

| $\mathcal{A}_7^j$ | Points of $\mathcal{D}$ | $\mathrm{Stab}(\mathcal{A}_7^j)$ |
|---|---|---|

| | | |
|---|---|---|
| $\mathcal{A}_7^1$ | 5,6,17 | **I** |
| $\mathcal{A}_7^2$ | 5,6,236 | $\mathbb{Z}_2$ |
| $\mathcal{A}_7^3$ | 5,6,336 | $\mathbb{Z}_3$ |
| $\mathcal{A}_7^4$ | 9,115,188 | $\mathbb{Z}_4$ |
| $\mathcal{A}_7^5$ | 9,98,256 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_7^6$ | 9,59,256 | $\mathfrak{S}_3$ |
| $\mathcal{A}_7^7$ | 26,285,339 | $\mathbb{Z}_6$ |
| $\mathcal{A}_7^8$ | 5,12,344 | $\mathbf{D}_6$ |
| $\mathcal{A}_7^9$ | 99,128,305 | $\mathfrak{S}_4$ |

In Table 3, $\mathrm{Stab}(\mathcal{A}_7^2) = \langle M_7^2 \rangle$, $\mathrm{Stab}(\mathcal{A}_7^3) = \langle M_7^3 \rangle$, $\mathrm{Stab}(\mathcal{A}_7^4) = \langle M_7^4 \rangle$ and $\mathrm{Stab}(\mathcal{A}_7^7) = \langle M_7^7 \rangle$, where

$$M_7^2 = \begin{bmatrix} -3 & 9 & -6 \\ 0 & 9 & -8 \\ 0 & 9 & -9 \end{bmatrix}, M_7^3 = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -8 & 0 \\ 0 & -8 & 8 \end{bmatrix}, M_7^4 = \begin{bmatrix} 0 & 6 & -5 \\ 2 & -8 & 0 \\ 0 & -2 & 0 \end{bmatrix} \text{ and } M_7^7 = \begin{bmatrix} 0 & 8 & -8 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The stabilizer group of the (7,3)-arc, $\mathcal{A}_7^5$ consists of the following 4 projective matrices:

$$M_{7,1}^5 = I, M_{7,2}^5 = \begin{bmatrix} 0 & 6 & -5 \\ -3 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}, M_{7,3}^5 = \begin{bmatrix} 0 & 6 & -5 \\ 0 & -1 & 0 \\ -4 & -5 & 0 \end{bmatrix} \text{ and } M_{7,4}^5 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & -4 \\ -4 & -5 & 0 \end{bmatrix}.$$

All the privious matrices has order 2 except the identity matrix. Hence, the stabilizer group of $\mathcal{A}_7^5$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The group $\mathrm{Stab}(\mathcal{A}_7^6)$ is non-abelian generating by the projective matrices $R$ and $S$, where

$$R=\begin{bmatrix}0&6&-5\\3&6&-9\\0&6&-6\end{bmatrix} \text{ and } S=\begin{bmatrix}1&0&0\\1&-1&0\\1&0&-1\end{bmatrix}.$$

In fact, $\mathrm{Stab}(\mathcal{A}_7^6)$ is isomorphic to $\mathfrak{S}_3$.

The stabilizer group of the (7,3)-arc, $\mathcal{A}_7^8$ is non-abelian consisting of the 12 projective matrices, and $\mathrm{Stab}(\mathcal{A}_7^8)=\langle R,S{:}\,R^6=S^2=I,SRS^{-1}=R^{-1}\rangle$, where

$$R=\begin{bmatrix}0&2&-1\\-7&3&0\\0&-8&0\end{bmatrix} \text{ and } S=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

Finally, the group of the (7,3)-arc, $\mathcal{A}_7^9$ has 24 projective matrices, and it is isomorphic to $\mathfrak{S}_4$. Moreover, $\mathrm{Stab}(\mathcal{A}_7^9)=\langle R,S\rangle$, where

$$R=\begin{bmatrix}1&0&0\\2&0&-1\\0&1&0\end{bmatrix} \text{ and } S=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

**3.2** The classification of $(\boldsymbol{v},\mathbf{3})$-arcs; $\boldsymbol{v}=\mathbf{8},\mathbf{9},\mathbf{10}$

In this subsection, the classification of $(v,3)$-arcs, where $v=8,9,10$, is established by classifying all such arcs up to stabilizer groups. By adding 4 preferred points of type 3 to $\mathcal{R}$, we get the following (8,3)-arcs as illustrated in Table 4.

**Table 4. Types of $(8,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}_8^j=\mathcal{R}\cup\mathcal{D}$**

| $\mathcal{A}_8^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_8^j$)** |
|---|---|---|
| $\mathcal{A}_8^1$ | 5, 6,7, 8 | **I** |
| $\mathcal{A}_8^2$ | 5, 6, 7, 330 | $\mathbb{Z}_2$ |
| $\mathcal{A}_8^3$ | 5, 6, 336, 265 | $\mathbb{Z}_3$ |

| | | |
|---|---|---|
| $\mathcal{A}_8^4$ | 5, 12, 344, 157 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_8^5$ | 5, 12, 344, 282 | $\mathbb{Z}_6$ |
| $\mathcal{A}_8^6$ | 25, 133, 40, 237 | $\mathbf{SL(2,3)}$ |

Let us explain the groups appeared in Table 4.

$\text{Stab}(\mathcal{A}_8^2) = \langle M_8^2 \rangle$, $\text{Stab}(\mathcal{A}_8^3) = \langle M_8^3 \rangle$, and $\text{Stab}(\mathcal{A}_8^5) = \langle M_8^5 \rangle$, where

$$M_8^2 = \begin{bmatrix} 3 & 1 & -3 \\ 0 & -5 & 5 \\ 0 & -7 & 5 \end{bmatrix}, M_8^3 = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -8 & 0 \\ 0 & -8 & 8 \end{bmatrix} \text{ and } M_8^5 = \begin{bmatrix} 0 & 2 & -1 \\ -7 & 3 & 0 \\ 0 & -8 & 0 \end{bmatrix}.$$

The stabilizer group of the (8,3)-arc, $\mathcal{A}_8^4$ consists of the following 4 projective matrices:

$$M_{8,1}^4 = I,\ M_{8,2}^4 = \begin{bmatrix} 0 & -1 & 2 \\ -7 & 0 & 3 \\ 0 & 0 & -8 \end{bmatrix}, M_{8,3}^4 = \begin{bmatrix} 9 & -4 & -4 \\ 9 & -3 & -6 \\ 9 & -6 & -3 \end{bmatrix} \text{ and } M_{8,4}^4 = \begin{bmatrix} -4 & 4 & 0 \\ -3 & 4 & 0 \\ -6 & 4 & 2 \end{bmatrix}.$$

All the privious matrices has order 2 except the identity matrix.

The stabilizer group of the (8,3)-arc, $\mathcal{A}_8^6$ is non-abelian consisting 24 projective matrices. In fact,

$\text{Stab}(\mathcal{A}_8^6) = \langle A, B, C : A^4 = C^3 = I, A^2 = B^2, BAB^{-1} = A^{-1}, CAC^{-1} = B, CBC^{-1} = AB \rangle$, where

$$A = \begin{bmatrix} 8 & -8 & 0 \\ 8 & 0 & -7 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -8 & 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 7 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 7 \end{bmatrix}.$$

Now, adding 5 preferred points of type 3 to $\mathcal{R}$, gives us (9,3)-arcs as shown in Table 5.

**Table 5. Types of $(9,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_9^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_9^j$ | **Points of $\mathcal{D}$** | **$\text{Stab}(\mathcal{A}_9^j)$** |
|---|---|---|

| | | |
|---|---|---|
| $\mathcal{A}_9^1$ | 5,6,50,52,376 | **I** |
| $\mathcal{A}_9^2$ | 5,41,43,247,363 | $\mathbb{Z}_2$ |
| $\mathcal{A}_9^3$ | 5,6,265,336,357 | $\mathbb{Z}_3$ |
| $\mathcal{A}_9^4$ | 5,12,157,303,344 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_9^5$ | 25,37,40,133,237 | $\mathbb{Z}_6$ |
| $\mathcal{A}_9^6$ | 5,12,282,327,344 | $\mathbf{D}_6$ |
| $\mathcal{A}_9^7$ | 25,40,133,237,247 | $\mathbf{G}_{216}$ |

In Table 5, $\mathrm{Stab}(\mathcal{A}_9^2) = \langle M_9^2 \rangle$, $\mathrm{Stab}(\mathcal{A}_9^3) = \langle M_9^3 \rangle$, and $\mathrm{Stab}(\mathcal{A}_9^5) = \langle M_9^5 \rangle$, where

$$M_9^2 = \begin{bmatrix} 9 & -8 & 0 \\ 9 & -9 & 0 \\ 7 & 8 & -3 \end{bmatrix}, M_9^3 = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -8 & 0 \\ 0 & -8 & 8 \end{bmatrix} \text{ and } M_9^5 = \begin{bmatrix} 1 & -1 & 0 \\ 8 & -1 & -7 \\ 1 & 7 & -7 \end{bmatrix}.$$

The stabilizer group of the (9,3)-arc, $\mathcal{A}_9^4$ consists of the 4 projective matrices, each one of them has of order 2 except the identity matrix:

$$M_{9,1}^4 = I,\ M_{9,2}^4 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -8 & 0 \\ -7 & 3 & 0 \end{bmatrix}, M_{9,3}^4 = \begin{bmatrix} 9 & -4 & -4 \\ 9 & -3 & -6 \\ 9 & -6 & -3 \end{bmatrix} \text{ and } M_{9,4}^4 = \begin{bmatrix} -4 & 0 & 4 \\ -6 & 2 & 4 \\ -3 & 0 & 4 \end{bmatrix}.$$

The stabilizer group of the (9,3)-arc, $\mathcal{A}_9^6$ is non-abelian consisting of the 12 projective matrices, and $\mathrm{Stab}(\mathcal{A}_9^6) = \langle R, S \colon R^6 = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} 0 & 2 & -1 \\ -7 & 3 & 0 \\ 0 & -8 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The group, $\mathrm{Stab}(\mathcal{A}_9^7)$, is non-abelian with 216 projective matrices. In fact,

(1) the identity matrix has order 1, say $M_1 = I$,

(2) there are 9 projective matrices of order 2, some of them are:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 0 & 8 & -7 \\ -1 & 0 & 1 \\ 0 & 0 & -7 \end{bmatrix}, \dots, M_{10},$$

(3) there are 80 projective matrices of order 3, some of them are:

$$M_{11} = \begin{bmatrix} 7 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 7 \end{bmatrix},\ M_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ 1 & 0 & 0 \end{bmatrix}, \dots, M_{90},$$

(4) there are 54 projective matrices of order 4, some of them are:

$$M_{91} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix},\ M_{92} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \dots, M_{144},$$

(5) there are 72 projective matrices of order 6, some of them are:

$$M_{145} = \begin{bmatrix} 0 & 0 & 1 \\ -8 & 0 & 8 \\ 0 & -1 & 1 \end{bmatrix},\ M_{146} = \begin{bmatrix} 1 & 0 & -1 \\ 8 & -8 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \dots, M_{216}.$$

Adding 6 preferred points of type 3 to $\mathcal{R}$, gives us six (10,3)-arcs up to stabilizer group as illustrated in Table 6.

**Table 6. Types of $(10,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}^j_{10} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}^j_{10}$ | **Points of $\mathcal{D}$** | **Stab$(\mathcal{A}^j_{10})$** |
|---|---|---|
| $\mathcal{A}^1_{10}$ | 5,50,52,376,374,6 | **I** |
| $\mathcal{A}^2_{10}$ | 5,12,344,282,375,52 | $\mathbb{Z}_2$ |
| $\mathcal{A}^3_{10}$ | 25,133,40,237,378,247 | $\mathbb{Z}_3$ |
| $\mathcal{A}^4_{10}$ | 25,37,40,133,237,146 | $\mathbb{Z}_4$ |
| $\mathcal{A}^5_{10}$ | 5,12,282,327,344,304 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}^6_{10}$ | 5,12,157,303,344,304 | $\mathbf{D}_6$ |

In Table 6, $\mathrm{Stab}(\mathcal{A}_{10}^2) = \langle M_{10}^2 \rangle$, $\mathrm{Stab}(\mathcal{A}_{10}^3) = \langle M_{10}^3 \rangle$, and $\mathrm{Stab}(\mathcal{A}_{10}^4) = \langle M_{10}^4 \rangle$, where

$$M_{10}^2 = \begin{bmatrix} 9 & -4 & -4 \\ 9 & -3 & -6 \\ 9 & -6 & -3 \end{bmatrix}, M_{10}^3 = \begin{bmatrix} 0 & 7 & -7 \\ 1 & 7 & -7 \\ 0 & 8 & -7 \end{bmatrix} \text{ and } M_{10}^4 = \begin{bmatrix} 0 & 0 & 1 \\ 7 & 0 & 1 \\ 0 & 8 & -7 \end{bmatrix}.$$

The stabilizer group of the $(10,3)$-arc, $\mathcal{A}_{10}^5$ consists of the 4 projective matrices, each one of them has of order 2 except the identity matrix:

$$M_{10,1}^5 = I,\ M_{10,2}^5 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -8 & 0 \\ -7 & 3 & 0 \end{bmatrix}, M_{10,3}^5 = \begin{bmatrix} 9 & -4 & -4 \\ 9 & -3 & -6 \\ 9 & -6 & -3 \end{bmatrix} \text{ and } M_{10,4}^5 = \begin{bmatrix} -4 & 0 & 4 \\ -6 & 2 & 4 \\ -3 & 0 & 4 \end{bmatrix}.$$

The stabilizer group of the $(10,3)$-arc, $\mathcal{A}_{10}^6$ is non-abelian consisting of the 12 projective matrices, and $\mathrm{Stab}(\mathcal{A}_{10}^6) = \langle R, S: R^6 = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} 0 & 2 & -1 \\ -7 & 3 & 0 \\ 0 & -8 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

### 3.3 The classification of $(\boldsymbol{v}, \mathbf{3})$-arcs; $\boldsymbol{v} = \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}$

In this subsection, the classification of $(v, 3)$-arcs, where $v = 11,12,13,14$, is established by classifying all such arcs up to stabilizer groups. By adding 7 preferred points

of type 3 to $\mathcal{R}$, we get the following $(11,3)$-arcs as shown in Table 7.

**Table 7. Types of $(11, 3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{11}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{11}^j$ | Points of $\mathcal{D}$ | $\mathrm{Stab}(\mathcal{A}_{11}^j)$ |
|---|---|---|
| $\mathcal{A}_{11}^1$ | 5,6,7,8,9,10,34 | I |
| $\mathcal{A}_{11}^2$ | 5,6,7,15,33,116,23 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{11}^3$ | 25,133,40,237,247,6,80 | $\mathbb{Z}_3$ |

| | | |
|---|---|---|
| $\mathcal{A}_{11}^4$ | 25,133,40,237,247,12,72 | $\mathbb{Z}_4$ |
| $\mathcal{A}_{11}^5$ | 5,12,344,157,303,37,228 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_{11}^6$ | 25,133,40,237,37,146,346 | $\mathbb{Z}_6$ |
| $\mathcal{A}_{11}^7$ | 5,6,236,224,281,31,60 | $\mathfrak{S}_3$ |

The stabilizer group, $\mathrm{Stab}(\mathcal{A}_{11}^1)$, is the trivial group. The group $\mathrm{Stab}(\mathcal{A}_{11}^2)$consists of two projective matrices, and hence it is isomorphic to $\mathbb{Z}_2$. Further, $\mathrm{Stab}(\mathcal{A}_{11}^3) = \langle M_{11}^3 \rangle$, $\mathrm{Stab}(\mathcal{A}_{11}^4) = \langle M_{11}^4 \rangle$ and $\mathrm{Stab}(\mathcal{A}_{11}^6) = \langle M_{11}^6 \rangle$, where

$$M_{11}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 8 & -8 \\ 0 & 1 & -8 \end{bmatrix}, M_{11}^4 = \begin{bmatrix} 1 & 8 & -8 \\ 0 & 8 & -8 \\ 0 & 1 & -8 \end{bmatrix} \text{ and } M_{11}^6 = \begin{bmatrix} -7 & 8 & 0 \\ -7 & 8 & -1 \\ -8 & 8 & 0 \end{bmatrix}.$$

The stabilizer group of $\mathcal{A}_{11}^5$ has 4 projective matrices, each one of them has of order 2 except the identity matrix, say

$$M_{11,1}^5 = I, M_{11,2}^5 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -8 & 0 \\ -7 & 3 & 0 \end{bmatrix}, M_{11,3}^5 = \begin{bmatrix} 9 & -4 & -4 \\ 9 & -3 & -6 \\ 9 & -6 & -3 \end{bmatrix} \text{ and } M_{11,4}^5 = \begin{bmatrix} -4 & 0 & 4 \\ -6 & 2 & 4 \\ -3 & 0 & 4 \end{bmatrix}.$$

The group $\mathrm{Stab}(\mathcal{A}_{11}^7)$ is non-abelian group of order 6. In fact, the generators of $\mathrm{Stab}(\mathcal{A}_{11}^7)$ are

$$R = \begin{bmatrix} -5 & 5 & 1 \\ 6 & 5 & 0 \\ 2 & 5 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -8 & 0 \\ 7 & 0 & 0 \end{bmatrix}.$$

Adding 8 preferred points of type 3 to the arc $\mathcal{R}$, implies four $(12,3)$-arcs up to stabilizer group as shown in Table 8.

**Table 8. Types of $(12,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{12}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{12}^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{12}^j$)** |
|---|---|---|

| | | |
|---|---|---|
| $\mathcal{A}_{12}^1$ | 5,6,7,8,9,10,34 | **I** |
| $\mathcal{A}_{12}^2$ | 5,6,7,15,33,116,23 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{12}^3$ | 25,133,40,237,247,6,80 | $\mathbb{Z}_3$ |
| $\mathcal{A}_{12}^4$ | 5,12,344,157,303,37,228 | $\mathbb{Z}_3 \times \mathbb{Z}_3$ |

The stabilizer groups Stab$(\mathcal{A}_{12}^2)$ and Stab$(\mathcal{A}_{12}^3)$ have 2 and 3 projective matrices, respectively. It follows that Stab$(\mathcal{A}_{12}^2) \cong \mathbb{Z}_2$ and Stab$(\mathcal{A}_{12}^3) \cong \mathbb{Z}_3$. Furthermore, The group Stab$(\mathcal{A}_{12}^4)$ is an abelian group with 9 projective matrices, and it has no element of order 9. Therefore, it is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Now, adding 9 preferred points of type 3 to the arc $\mathcal{R}$, implies four $(13,3)$-arcs as shown in

Table 9.

**Table 9. Types of $(13,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{13}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{13}^j$ | **Points of $\mathcal{D}$** | **Stab$(\mathcal{A}_{13}^j)$** |
|---|---|---|
| $\mathcal{A}_{13}^1$ | 5,7,15,33,45,54,59,245,315 | **I** |
| $\mathcal{A}_{13}^2$ | 5,12,344,157,303,37,304,46,25 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{13}^3$ | 5,6,7,8,72,132,133,40,146 | $\mathbb{Z}_3$ |
| $\mathcal{A}_{13}^4$ | 5,12,344,157,303,37,304,46,85 | $\mathfrak{S}_3$ |

The stabilizer groups Stab$(\mathcal{A}_{13}^2)$ and Stab$(\mathcal{A}_{13}^3)$ have 2 and 3 projective matrices, respectively. It follows that Stab$(\mathcal{A}_{13}^2) \cong \mathbb{Z}_2$ and Stab$(\mathcal{A}_{13}^3) \cong \mathbb{Z}_3$. Furthermore, The group Stab$(\mathcal{A}_{13}^4)$ is non-abelian group with 6 projective matrices. Therefore, it is isomorphic to $\mathfrak{S}_3$.

There are 3 types of (14,3)-arcs up to stabilizer group as shown in Table 10. These arcs can be constructed by adding 10 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 10. Types of $(14,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}^j_{14} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}^j_{14}$ | Points of $\mathcal{D}$ | Stab$(\mathcal{A}^j_{14})$ |
|---|---|---|
| $\mathcal{A}^1_{14}$ | 5,6,7,50,52,204,209,310,312,376 | I |
| $\mathcal{A}^2_{14}$ | 5,12,157,289,298,303,304,344,356,367 | $\mathbb{Z}_2$ |
| $\mathcal{A}^3_{14}$ | 25,133,40,237,247,6,80,334,47,171 | $\mathbb{Z}_3$ |

The stabilizer group Stab$(\mathcal{A}^1_{14})$ is the trivial group. Further, the stabilizer groups Stab$(\mathcal{A}^2_{14})$ and Stab$(\mathcal{A}^3_{14})$ have 2 and 3 projective matrices, respectively. It follows that Stab$(\mathcal{A}^2_{14}) \cong \mathbb{Z}_2$ and Stab$(\mathcal{A}^3_{14}) \cong \mathbb{Z}_3$.

**3.4** The classification of $(\boldsymbol{\nu}, \mathbf{3})$-arcs; $\boldsymbol{\nu} = \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}$

In this subsection, the classification of $(\nu, 3)$-arcs, where $\nu = 15, 16,17,18,19,20$, is established by classifying all such arcs up to stabilizer groups. By adding 11 preferred points of type 3 to $\mathcal{R}$, we get the following (15,3)-arcs as shown in Table 11.

**Table 11. Types of $(15,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}^j_{15} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}^j_{15}$ | Points of $\mathcal{D}$ | Stab$(\mathcal{A}^j_{15})$ |
|---|---|---|
| $\mathcal{A}^1_{15}$ | 5,6,7,15,33,45,54,59,245,315,378 | I |
| $\mathcal{A}^2_{15}$ | 5,12,37,46,85,157,256,303,304,344,371 | $\mathbb{Z}_2$ |
| $\mathcal{A}^3_{15}$ | 6,25,40,47,80,133,171,237,247,315,334 | $\mathbb{Z}_3 \times \mathbb{Z}_3$ |

All the groups in Table 11 are abelian of order 1, 2 and 9. Therefore, the stabilizer group, $\mathrm{Stab}(\mathcal{A}_{15}^{1})$, is the trivial group. The group $\mathrm{Stab}(\mathcal{A}_{15}^{2}) \cong \mathbb{Z}_2$ and $\mathrm{Stab}(\mathcal{A}_{15}^{3}) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

There are 3 types of (16,3)-arcs up to stabilizer group as shown in Table 12. These arcs can be constructed by adding 12 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 12. Types of $(16,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^{2}$; $\mathcal{A}_{16}^{j} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{16}^{j}$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{16}^{j}$)** |
|---|---|---|
| $\mathcal{A}_{16}^{1}$ | 5,7,15,33,45,54,59,245,315,378,316,299 | **I** |
| $\mathcal{A}_{16}^{2}$ | 5,12,22,157,289,298,301,303,304,344,356,367 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{16}^{3}$ | 6,25,40,47,80,115, 133,171,237,247,315,334 | $\mathbb{Z}_3$ |

All the groups in Table 12 are abelian of order 1, 2 and 3. Therefore, the stabilizer group, $\mathrm{Stab}(\mathcal{A}_{16}^{1})$, is the trivial group. The group $\mathrm{Stab}(\mathcal{A}_{16}^{2}) \cong \mathbb{Z}_2$ and $\mathrm{Stab}(\mathcal{A}_{16}^{3}) \cong \mathbb{Z}_3$.

There are 3 types of (17,3)-arcs up to stabilizer group. These arcs can be established by adding 13 preferred points of type 3 to the arc $\mathcal{R}$. Table 13 illustrates all these arcs.

**Table 13. Types of $(17,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^{2}$; $\mathcal{A}_{17}^{j} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{17}^{j}$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{17}^{j}$)** |
|---|---|---|
| $\mathcal{A}_{17}^{1}$ | 5,7,15,33,45,54,59, 245,315,378,316,299,320 | **I** |
| $\mathcal{A}_{17}^{2}$ | 5,7,15,33,45,54,59, 245,315,378,316,299,287 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{17}^{3}$ | 6,25,40,47,80,115,133, 171,203,237,247,315,334 | $\mathbb{Z}_3$ |

Similarly, all the groups in Table 13 are abelian of orders 1, 2 and 3. It follows that the stabilizer group, $\mathrm{Stab}(\mathcal{A}_{17}^{1})$, is the trivial group. The group $\mathrm{Stab}(\mathcal{A}_{17}^{2}) \cong \mathbb{Z}_2$ and $\mathrm{Stab}(\mathcal{A}_{17}^{3}) \cong \mathbb{Z}_3$.

There are 4 types of (18,3)-arcs up to stabilizer group. These arcs can be established by adding 14 preferred points of type 3 to the arc $\mathcal{R}$. Table 14 illustrates all these arcs.

**Table 14. Types of $(18,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}^j_{18} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}^j_{18}$ | Points of $\mathcal{D}$ | Stab$(\mathcal{A}^j_{18})$ |
|---|---|---|
| $\mathcal{A}^1_{18}$ | 5,7,15,33,45,54,59, 245,315,378,316,299,320,254 | I |
| $\mathcal{A}^2_{18}$ | 5,7,15,33,45,54,59, 245,315,378,316,299,287,260 | $\mathbb{Z}_2$ |
| $\mathcal{A}^3_{18}$ | 6,25,40,47,80,133,171, 237,247,251,315,334,355,356 | $\mathbb{Z}_3$ |
| $\mathcal{A}^4_{18}$ | 6,25,40,47,80,115,133, 171,203,237,247,261,315,334 | $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ |

The groups in Table 14 are abelian of orders 1, 2, 4 and 27. The stabilizer group, Stab$(\mathcal{A}^1_{18})$, is the trivial group. The group Stab$(\mathcal{A}^2_{18}) \cong \mathbb{Z}_2$ and Stab$(\mathcal{A}^3_{18}) \cong \mathbb{Z}_3$. Every element in Stab$(\mathcal{A}^4_{18})$ has order 3, so it is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

There are 3 types of (19,3)-arcs up to stabilizer group as shown in Table 15. These arcs can be constructed by adding 15 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 15. Types of $(19,3)$-arcs up to stabilizer groups in $\mathbb{P}^2_{19}$; $\mathcal{A}^j_{19} = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}^j_{19}$ | Points of $\mathcal{D}$ | Stab$(\mathcal{A}^j_{19})$ |
|---|---|---|
| $\mathcal{A}^1_{19}$ | 5,7,15,33,45,54,59,245, 315,378,316,299,320,254,196 | I |
| $\mathcal{A}^2_{19}$ | 5,7,15,33,45,54,59,94, 245,260,287,299,315,316,378 | $\mathbb{Z}_2$ |
| $\mathcal{A}^3_{19}$ | 6,25,40,47,80,115,133,171, 237,247,251,315,334,355,356 | $\mathbb{Z}_3$ |

Again, all the groups in Table 15 are abelian of orders 1, 2, and 3. Hence, the stabilizer group, Stab$(\mathcal{A}^1_{19})$, is the trivial group, and the group Stab$(\mathcal{A}^2_{19}) \cong \mathbb{Z}_2$ and Stab$(\mathcal{A}^3_{19}) \cong \mathbb{Z}_3$.

There are 2 types of (20,3)-arcs up to stabilizer group as shown in Table 16. These arcs can be constructed by adding 16 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 16. Types of $(20, 3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{20}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{20}^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{20}^j$)** |
|---|---|---|
| $\mathcal{A}_{20}^1$ | 5,7,15,33,45,54,59,245, 315,378,316,299,320,254,196,244 | **I** |
| $\mathcal{A}_{20}^2$ | 6,25,40,47,80,115,133,171, 203,237,247,251,315,334,355,356 | $\mathbb{Z}_3$ |

All the groups in Table 16 are abelian of orders 1 and 3. Hence, the stabilizer group, Stab($\mathcal{A}_{20}^1$), is the trivial group, and the group, Stab($\mathcal{A}_{20}^2$) $\cong \mathbb{Z}_3$.

**3.5** The classification of $(\nu, 3)$-arcs; $\boldsymbol{\nu = 21, 22, 23, 24, 25, 26, 27, 28, 29, 30}$

In this subsection, we establish the distinct $(\nu, 3)$-arcs up to their stabilizer groups, where $\nu = 21, \ldots, 30$. By similar arguments in the previous subsections, if we adding 17 preferred

points of type 3 to the arc $\mathcal{R}$, we get the following (21,3)-arcs as shown in Table 17.

**Table 17. Types of $(21, 3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{21}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{21}^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{21}^j$)** |
|---|---|---|
| $\mathcal{A}_{21}^1$ | 5,7,15,33,45,54,59,245,315, 378,316,299,320,254,196,244,309 | **I** |
| $\mathcal{A}_{21}^2$ | 6,25,40,47,80,115,133,171,203, 237,247,251,261,315,334,355,356 | $\mathbb{Z}_3$ |
| $\mathcal{A}_{21}^3$ | 12,16,38,43,66,110,171,217,253, 282,291,298,327,344,349,367,378 | $\mathbf{D}_{18}$ |

| | | |
|---|---|---|
| $\mathcal{A}_{21}^4$ | 12,16,38,43,66,110,171,217,253, 282,291,298,327,344,349,367,381 | $\mathbf{D}_{20}$ |

The stabilizer group, $\text{Stab}(\mathcal{A}_{21}^1)$, is the trivial group, and the group $\text{Stab}(\mathcal{A}_{21}^2) \cong \mathbb{Z}_3$ because it has three projective matrices**.** The stabilizer group of the (21,3)-arc, $\mathcal{A}_{21}^3$is non-abelian consisting of the 36 projective matrices, and $\text{Stab}(\mathcal{A}_{21}^3) = \langle R, S: R^{18} = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} -4 & 9 & -4 \\ -3 & -3 & 6 \\ -6 & -2 & 8 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 6 & -5 \\ 9 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Furthermore, The stabilizer group of the (21,3)-arc, $\mathcal{A}_{21}^4$is non-abelian consisting of the 40 projective matrices, and $\text{Stab}(\mathcal{A}_{21}^4) = \langle R, S: R^{20} = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} 8 & 7 & 5 \\ 9 & -1 & -9 \\ -8 & 5 & -3 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 9 & 0 \\ 5 & 0 & 0 \end{bmatrix}.$$

There are 4 types of (22,3)-arcs up to stabilizer group as shown in Table 18. These arcs can be constructed by adding 18 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 18. Types of $(22,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{22}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{22}^j$ | **Points of $\mathcal{D}$** | $\text{Stab}(\mathcal{A}_{22}^j)$ |
|---|---|---|
| $\mathcal{A}_{22}^1$ | 12,16,38,43,66,110,111,217,253,282,291,298,327,344,349,367,5,368 | **I** |
| $\mathcal{A}_{22}^2$ | 12,16,38,43,66,110,111,217,253,282,291,298,327,344,349,367,5,171 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{22}^3$ | 12,16,38,43,66,110,171,217,253,282,291,298,327,344,349,367,378,373 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_{22}^4$ | 12,16,38,39,43,66,110,171,217,253,282,291,298,327,344,349,367,378 | $\mathbf{D}_4$ |

In Table 18, $\text{Stab}(\mathcal{A}_{22}^2)$ is isomorphic to $\mathbb{Z}_2$ because it has two projective matrices. The group $\text{Stab}(\mathcal{A}_{22}^3)$ has 4 projective matrices, each one of them has order 2 except the identity, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, The stabilizer group of the (22,3)-arc, $\mathcal{A}_{22}^4$is non-abelian consisting of the 8 projective matrices, and $\mathrm{Stab}(\mathcal{A}_{22}^4) = \langle R, S : R^4 = S^2 = I, SRS^{-1} = R^{-1} \rangle$, where

$$R = \begin{bmatrix} -3 & 5 & -1 \\ 2 & 6 & -8 \\ -9 & 7 & 2 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 8 & 0 \\ -6 & 4 & 0 \end{bmatrix}.$$

There are 4 types of (23,3)-arcs up to stabilizer group as shown in Table 19. These arcs can be constructed by adding 19 preferred points of type 3 to the arc $\mathcal{R}$.

**Table 19. Types of $(23,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{23}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{23}^j$ | Points of $\mathcal{D}$ | $\mathrm{Stab}(\mathcal{A}_{23}^j)$ |
|---|---|---|
| $\mathcal{A}_{23}^1$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,368,171 | I |
| $\mathcal{A}_{23}^2$ | 12,16,38,43,66,110,171,217,253,282,291, 298,327,344,349,367,378,373,361 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{23}^3$ | 12,16,38,39,43,66,75,110,171,217,253,282,291, 298,327,344,349,367,378 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_{23}^4$ | 5,12,16,38,43,66,110,111,156,171,217,253,282,291, 298,327,344,349,367 | $\mathfrak{S}_3$ |

In Table 19, $\mathrm{Stab}(\mathcal{A}_{23}^2)$ is isomorphic to $\mathbb{Z}_2$ because it has two projective matrices. The group $\mathrm{Stab}(\mathcal{A}_{23}^3)$ has 4 projective matrices, each one of them has order 2 except the identity, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, The stabilizer group of the (23,3)-arc, $\mathcal{A}_{23}^4$is non-abelian with 6 projective matrices, and $\mathrm{Stab}(\mathcal{A}_{23}^4) = \langle R, S \rangle$, where

$$R = \begin{bmatrix} 6 & -3 & -2 \\ 9 & -6 & 0 \\ -5 & -2 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 9 & 0 & -8 \\ 5 & -5 & 8 \\ 7 & 0 & -9 \end{bmatrix}.$$

By similar arguments, if we adding 20 preferred points of type 3 to the arc $\mathcal{R}$, we get the following (24,3)-arcs as shown in Table 20.

**Table 20. Types of $(24,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{24}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{24}^j$ | **Points of $\mathcal{D}$** | **Stab($\mathcal{A}_{24}^j$)** |
|---|---|---|
| $\mathcal{A}_{24}^1$ | 12,16,38,43,66,110,171, 217,253,282,291, 298,327,344, 349,367,378,373,361,226 | **I** |
| $\mathcal{A}_{24}^2$ | 12,16,28,38,39,43,66, 75,110,171,217,253,282,291, 298,327,344,349,367,378 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{24}^3$ | 12,16,38,43,66,102,110, 171,217,253,282,291, 298,327, 344,349,361,367,373,378 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |

In Table 20, $\mathrm{Stab}(\mathcal{A}_{24}^2)$ is isomorphic to $\mathbb{Z}_2$ because it has two projective matrices. The group $\mathrm{Stab}(\mathcal{A}_{24}^3)$ has 4 projective matrices, each one of them has order 2 except the identity, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Again, if we add 21 preferred points of type 3 to the arc $\mathcal{R}$, we get the three types of (25,3)-arcs up to stabilizer group as shown in Table 21.

**Table 21. Types of $(25,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^2$; $\mathcal{A}_{25}^j = \mathcal{R} \cup \mathcal{D}$**

| $\mathcal{A}_{25}^{j}$ | **Points of $\mathcal{D}$** | **Stab**$(\mathcal{A}_{25}^{j})$ |
|---|---|---|
| $\mathcal{A}_{25}^{1}$ | 12,16,38,43,66,110,171, 217,253,282,291, 298,327,344, 349,367,378,373,361,226 | **I** |
| $\mathcal{A}_{25}^{2}$ | 12,16,28,38,39,43,66, 75,110,171,217,253,282,291, 298,327,344,349,367,378 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{25}^{3}$ | 12,16,38,43,66,102,110, 171,217,253,282,291, 298,327, 344,349,361,367,373,378 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |

In Table 21, Stab$(\mathcal{A}_{25}^{2})$ is isomorphic to $\mathbb{Z}_2$ because it has two projective matrices. The group Stab$(\mathcal{A}_{25}^{3})$ has 4 projective matrices, each one of them has order 2 except the identity, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, adding 22, 23, 24 and 25 preferred points of type 3 to the arc $\mathcal{R}$, will give us two (26,3)-arcs, three (27,3)-arcs, two (28,3)-arcs and two (29,3)-arcs, respectively as shown in Table 22.

**Table 22. Types of $(26,3),(27,3),(28,3),(29,3)$-arcs up to stabilizer groups in $\mathbb{P}_{19}^{2}$; $\mathcal{A}_{k}^{j} = \mathcal{R} \cup \mathcal{D}, k = 26,27,28,29$**

| $\mathcal{A}_{k}^{j}$ | **Points of $\mathcal{D}$** | **Stab**$(\mathcal{A}_{k}^{j})$ |
|---|---|---|
| $\mathcal{A}_{26}^{1}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,368,319,171,39,337 | **I** |
| $\mathcal{A}_{26}^{2}$ | 12,16,28,38,39,43,66,75,108,110,171,217,253,282,291, | $\mathbb{Z}_2$ |

| | | |
|---|---|---|
| | 296,298,327,344,349,367,378 | |
| $\mathcal{A}_{27}^{1}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,379,314,266,286,274,171 | **I** |
| $\mathcal{A}_{27}^{2}$ | 12,16,28,38,39,43,66,75,108,110,171,217,253,282,291, 296,298,322,327,344,349,367,378 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{27}^{3}$ | 12,16,28,38,39,43,66,75,108,110,171,217,253,282,291, 296,298,310,327,344,349,367,378 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathcal{A}_{28}^{1}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,379,314,266,286,274,279,171 | **I** |
| $\mathcal{A}_{28}^{2}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,379,314,266,286,274,351,171 | $\mathbb{Z}_2$ |
| $\mathcal{A}_{29}^{1}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,379,314,266,171,155,274,279,286 | **I** |
| $\mathcal{A}_{29}^{2}$ | 12,16,38,43,66,110,111,217,253,282,291, 298,327,344,349,367,5,379,314,266,286,274,279,171,351 | $\mathbb{Z}_2$ |

In Table 22, The stabilizer groups, $\mathrm{Stab}(\mathcal{A}_{26}^{1})$, $\mathrm{Stab}(\mathcal{A}_{27}^{1})$, $\mathrm{Stab}(\mathcal{A}_{28}^{1})$ and $\mathrm{Stab}(\mathcal{A}_{29}^{1})$ are the trivial groups, because they have only one projective matrix. The stabilizer groups, $\mathrm{Stab}(\mathcal{A}_{26}^{2})$, $\mathrm{Stab}(\mathcal{A}_{27}^{2})$, $\mathrm{Stab}(\mathcal{A}_{28}^{2})$ and $\mathrm{Stab}(\mathcal{A}_{29}^{2})$ are isomorphic to $\mathbb{Z}_2$ because it has two projective matrices. The group $\mathrm{Stab}(\mathcal{A}_{27}^{3})$ has 4 projective matrices, each one of them has order 2 except the identity, so it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Finally, if we add 26 preferred points of type 3 to the arc $\mathcal{R}$, we have

only (30,3)-arc, say $\mathcal{A}_{30} = \mathcal{R} \cup \mathcal{D}$, where

$$\mathcal{D} = \begin{Bmatrix} 5,9,12,16,17,38,43,66,110,171,189,217,253,254, \\ 282,284,291,298,307,318,327,344,349,367,381,358 \end{Bmatrix}.$$

The stabilizer group of the (30,3)-arc, $\mathcal{A}_{30}$ is isomorphic to $\mathbb{Z}_2$. In fact, $\text{Stab}(\mathcal{A}_{30}) = \langle M \rangle$, where

$$M = \begin{bmatrix} 6 & 6 & 8 \\ -7 & -9 & 9 \\ 4 & 7 & -8 \end{bmatrix}.$$

According to the arguments in sections 3.1, 3.2, 3.3, 3.4 and 3.5, we present the following theorem:

**Theorem.** A In the projective plane of order 19, $\mathbb{P}^2_{19}$, we have:

1. There are 9 projectively inquivalent (7,3)-arcs up to stabilizer groups.
2. There are 8 projectively inquivalent (6,3)-arcs up to stabilizer groups.
3. For $k = 7{,}11$, there are 7 projectively inquivalent $(k, 3)$-arcs up to stabilizer groups.
4. For $k = 8{,}10$, there are 6 projectively inquivalent $(k, 3)$-arcs up to stabilizer groups.
5. For $k = 5{,}12{,}13{,}18{,}21{,}22, 23$, there are 4 projectively inquivalent $(k, 3)$-arcs up to stabilizer groups.
6. For $k = 14{,}15{,}16{,}17{,}19{,}24, 25{,}27$, there are 3 projectively inquivalent $(k, 3)$-arcs up to stabilizer groups.
7. For $k = 20{,}26{,}28, 29$, there are 2 projectively inquivalent $(k, 3)$-arcs up to stabilizer groups.
8. There is only one projectively inquivalent (30,3)-arcs up to stabilizer groups.

## Conclusions

In summary, the large size of complete $(k, 3)$-arcs which is 30 is cunstructed by classify all

$(k, 3)$-arcs according to their stabilizer groups. Additionally, the are two projectively inequivalent $(k, 3)$-arcs for an integer $k$, where $k = 20{,}26{,}28, 29$. Moreove, we find three projectively inquivalent $(k, 3)$-arcs up to stabilizer groups for the value of $k$, where $k = 14{,}15{,}16{,}17{,}19{,}24, 25{,}27,$. As well as, for $k = 8{,}10$, there are six projectively inquivalent $(k, 3)$-arcs up to stabilizer groups. Furthermore, there are seve type of projectively inquivalent $(k, 3)$-arcs up are found for $k = 7{,}11$.

xxope (FESEM).

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## تصنيف أقواس الدرجة الثالثة في PG(2,19) حسب زمر التثبيت

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**المستخلص**

القوس من النمط $(n,3)$، $\mathcal{K}$ في المستوى الإسقاطي $PG(2,q)$ ذو الحجم $n$ والدرجة الثالثة عبارة عن مجموعة من النقاط $n$ بحيث يلتقي بها كل خط في المستوى في أقل من أو يساوي ثلاث نقاط، وكذلك القوس $\mathcal{K}$ تكون كاملة إذا لم تكن موجودة في قوس من النمط $(n+1,3)$. في هذا البحث تم تقديم تصنيف الأقواس من الدرجة الثالثة في $PG(2,q)$ بالتفصيل حسب زمر التثبيت الخاصة بها. الدافع للعمل في المستوى الإسقاطي للطلب 19 ذو شقين. أولاً، حجم القوس الأكبر $(n,3)$ غير معروف. ثانيًا، عدد الأقواس $(n,3)$ أعلى بكثير في المستوى الإسقاطي من الرتبة 19 مما هو عليه في المستوى الإسقاطي من الرتبة q لاجل $q < 19$، مما يعطي عددًا كبيرًا من الاقواس ذو النمط $(n,3)$ للدراسة.