Some results on nil-injective rings

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Let R be a ring. A right R-module is called nil-injective if for any element ω is belong to the set of nilpotent elements, and any right R-homomorphism can be extended to $R \to M$. If R_R is nil-injective, then R is called a right nil-injective ring. A right R-module is called Wnil-injective if for each non-zero nilpotent element ω of R, there exists a positive integer n such that $\omega^n \neq 0$ that right R-homomorphism $f: \omega^n R \to M$ can be extended to $R \to M$. If R_R is right Wnil-injective, then R is called a right Wnil-injective ring. In the present work, we discuss some characterizations and properties of right nil-injective and Wnil-injective rings.

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1. Introduction

In this article, R is an associative ring with identity, and All R-modules are unital. We denote $r_R\ (\omega)$ and $l_R\ (\omega)$ to the right annihilator and the left annihilator of ω, respectively. The set of nilpotent elements, the set of unit elements, the set of right singular elements and the Jacobson radical of R are denoted by $N(R), U(R), Z(R_R)$, and $J(R)$, respectively. Also, by $\mathbb{Z}_n$ and $\mathbb{Z}$, we mean the set of integers modulo n and integer numbers, respectively. In addition, an R-module M is called p-injective if for any principal right ideal I of R and any right R-homomorphism $g: I \to M$, there exists $\Upsilon \in M$ such that $g(\upsilon) = \upsilon\Upsilon$, for all υ in I, which was first introduced by Ming in [9]. In [10] also, Yue Chi Ming generalized p-injective, which is np-injective. A right R-module M is called right np-injective if for any $\omega \notin N(R)$ and any R-homomorphism $f: \omega R \to M$ can be extended to $R \to M$, or equivalently, for any $\omega \notin N(R)$ and any R-homomorphism $f: \omega R \to M$, there exists $m \in M$ such that $f(x) = m\Upsilon$, for all $\Upsilon \in \omega R$. So, the ring R is called right np-injective if R_R is np-injective. Wei and Chen defined weakly np-injective in [7]. A right R-module M is called weakly np-injective if for any $\omega \notin N(R)$, there exists a positive integer n such that $\omega^n \neq 0$ and any right R-homomorphism $f: \omega^n R \to M$ can be extended to $R \to M$. Or equivalently, for any $\omega \notin N(R)$, there exists a positive integer n such that $\omega^n \neq 0$ and any R-homomorphism $f: \omega^n R \to M$ there exists $m \in M$ such that $f(x) = m\Upsilon$, for all $\Upsilon \in \omega^n R$. If R_R is weakly np-injective, then R is a right weakly np-injective ring. It is easy to check that every right np-injective module is right weakly np-injective. Wei and Chen [6] generalized p-injective to nil-injective. They have defined that a right R-module M is called nil-injective, if for any $\omega \in N(R)$ and any R-homomorphism $f : \omega R \to M$ can be extended to $f: R \to M$, or equivalently, for any $\omega \in N(R)$ and any R-homomorphism $f : \omega R \to M$ there exists $m \in M$ such that $f(x) = m\Upsilon$, for all $\Upsilon \in \omega R$. So, the ring R is called right nil-injective if R_R is nil-injective. A right R-module M is called Wnil-injective if for any $0 \neq \omega \in N(R)$, there exists a positive integer n such that $\omega^n \neq 0$ and any right R-homomorphism $f: a^n R \to M$ can be extended to $R \to M$. Or equivalently, for any $\omega \in N(R)$, there exists a positive integer n such that $\omega^n \neq 0$ and any R-homomorphism $f: a^n R \to M$ there exists $m \in M$ such that $f(x) = m\Upsilon$ for all $\Upsilon \in \omega^n R$ [6]. A ring R is called semiprimitive ring if $J(R) = 0$ [1]. We found that if R is right continuous ring and $\mathrm{R}_{\mathrm{R/J(R)}}$ is nil injective ring, then R is semiprimitive. In the matrix ring, If $\mathrm{M_n}(R)$ is a right Wnil-injective ring, for some $\mathrm{n} \geq 2$, then R is a right nil-injective ring.

2. Nil-Injective Rings

In this section, we consider some examples and primary results about nil-injective. Wei and Chen [6] are poved that a ring R is a right nil-injective if and only if $l_R(r_R(\omega)) = R\omega$, for every $\omega \in N(R)$. We found some non-tivial examples of nil-injective rings via those theorem. Recall that if the ring of scalars R is commutative, then for all $\kappa \in R$ and $\mu \in M$, we have $\kappa\mu = \mu\kappa$. Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(\kappa, \mu): \kappa \in R, \mu \in M\}$ with addition defined componentwise and multiplication defined by $(\kappa, \mu)(\nu, \chi) = (\kappa\nu, \kappa\chi + \mu\nu)$ [5]. So, we obtain that for any $\kappa \in N(R)$, for $(\kappa, \mu) \in S' = R \propto M$, there exist $n \in \mathbb{Z}^+$ such that $\kappa^n = 0$, then $(\kappa, \mu)^{n+1} = (\kappa^{n+1}, (n+1)\kappa^n\mu) = (0,0)$, for every $\kappa \in N(R)$ and for every $\mu \in M$. Thus, the set of nilpotent elements in $R \propto M$ is given by: $N(R \propto M) = \{(\kappa, \mu) | \kappa \in N(R) \ and \ \mu \in M\}$. In addition, we found some examples which are not p-injective rings but they are nil-injective rings:

Example 2.1 Let $\mathrm{S} = \mathrm{R} \propto \mathrm{M} = \mathbb{Z} \propto \mathbb{Z}_4 = \{(\kappa, \mu) | \kappa \in \mathbb{Z} \text{ and } \mu \in \mathbb{Z}_4\}$ be a ring with addition defined componentwise and multiplication defined by $(\kappa, \mu)(\nu, \chi) = (\kappa\nu, \kappa\chi + \mu\nu)$. Now, $\mathrm{N(S)} = \{(0,0), (0,1), (0,2), (0,3)\}$. Firstly, $\mathrm{l_S(r_S((0,0)))} = \{(0,0)\} = \mathrm{S}(0,0)$ and $\mathrm{l_S(r_S((0,1)))} = \{(0, \mu) | \mu \in \mathbb{Z}_4\} = \mathrm{S}(0,1)$. Secondly, $\mathrm{r_S}((0,2)) = \{(\kappa, \mu) | \kappa \in <2> \text{ and } \mu \in \mathbb{Z}_4\}$. So, $\mathrm{l_S(r_S((0,2)))} = \{(0, 2\mu) | \mu \in \mathbb{Z}_4\} = \mathrm{S}(0,2)$. Thirdly, $\mathrm{r_S}((0,3)) = \{(\kappa, \mu) | \kappa \in <2> \text{ and } \mu \in \mathbb{Z}_4\}$. So, $\mathrm{l_S(r_S((0,3)))} = \{(0, \mu) | \mu \in \mathbb{Z}_4\} = \mathrm{S}(0,3)$. Thus, S is right nil-injective ring. But, S is not right p-injective ring because $(2,0) \in \mathrm{S}$. Then, $\mathrm{r_S}((2,0)) = \{(0, 2\mu) | \mu \in \mathbb{Z}_4\}$. So, $\mathrm{l_S(r_S((2,0)))} = \{(\kappa, \mu) | \kappa \in <2> \text{ and } \mu \in \mathbb{Z}_4\}$, but $\mathrm{S}(2,0) = \{(\kappa, 2\mu) | \kappa \in <2> \text{ and } \mu \in \mathbb{Z}_4\}$. Thus, $\mathrm{l_S(r_S((2,0)))} \neq \mathrm{S}(2,0)$. Hence, S is not right p-injective ring.

Example 2.2 Let $\mathrm{S} = \mathbb{Z} \oplus \mathbb{Z}_4 = \{(\kappa, \mu) | \kappa \in \mathbb{Z} \text{ and } \mu \in \mathbb{Z}_4\}$ be an external direct sum of $\mathbb{Z}$ and $\mathbb{Z}_4$ with standard addition and multiplication. Since $\mathrm{N(S)} = \{(0,0), (0,2)\}$. Firstly, $\mathrm{l_S(r_S((0,0)))} = \{(0,0)\} = \mathrm{S}(0,0)$. Secondly, $\mathrm{r_S}\big((0,2)\big) = \{(\kappa, \mu) | (0,2)(\kappa, \mu) = (0,0), \kappa \in \mathbb{Z} \text{ and } \mu \in \mathbb{Z}_4\} = \big\{(\kappa, \mu) |, \kappa \in \mathbb{Z} \text{ and } \mu \in \mathrm{r}_{\mathbb{Z}_4}(2)\big\}$. Then, $\mathrm{l_S(r_S((0,2)))} = \{(0, \beta) | \beta \in \langle 2 \rangle_4\} = \mathrm{S}(0,2)$. Thus, S is right nil-injective ring. But, S is not right p-injective ring because $(3,0) \in \mathrm{S}$. Then, $\mathrm{r_S}((3,0)) = \{(0, \mu) | \mu \in \mathbb{Z}_4\}$. So, $\mathrm{l_S(r_S((3,0)))} = \{(\kappa, 0) | \kappa \in \mathbb{Z}\}$, but $\mathrm{S}(3,0) = \{(3\kappa, 0) | \kappa \in \mathbb{Z}\}$. Thus, $\mathrm{l_S(r_S((3,0)))} \neq \mathrm{S}(3,0)$. Hence, S is not right p-injective ring.

Proposition 2.3 If $S = R \propto M = \mathbb{Z} \propto \mathbb{Z}_n$. Then, S_S is right nil-injective ring.

Proof. Let $S = \mathbb{Z} \propto \mathbb{Z}_n = \{(\kappa, \bar{\mu}) | \kappa \in \mathbb{Z} \text{ and } \bar{\mu} \in \mathbb{Z}_n\}$. Then, $N(S) = \{(0, \bar{\mu}) | \bar{\mu} \in \mathbb{Z}_n\}$. So,

$$r_S((0,\bar{\mu})) = \{(0,\bar{\mu})(x,\bar{y}) = (0,\bar{0}) | x \in \mathbb{Z} \text{ and } \bar{y} \in \mathbb{Z}_n\} = \{(x,\bar{y}) | \bar{\mu}x = \bar{0}, x \in \mathbb{Z} \text{ and } \bar{\mu} \in \mathbb{Z}_n\}$$

$$= \{(mp, \bar{y}) \in S | \text{ where } n = p\mu, \text{for all } m \in \mathbb{Z} \text{ and for some } \mu, n, p \in \mathbb{Z}\}.$$

We have two cases for find $l_S(r_S((0,\bar{\mu})))$. Firstly, if $\bar{\mu}$ is non-zero divisor, then $\bar{\mu}$ is unit. There is nothig to prove. Secondly, if $\bar{\mu}$ is zero divisor, $l_S(r_S((0,\bar{\mu}))) = \{(a,\bar{b}) | (a,\bar{b})(mp,\bar{y}) = (0,\bar{0}) | \text{where } n = p\mu, \text{for all } (mp,\bar{y}) \in S, \text{for all } m \in \mathbb{Z} \text{ and for some } \mu, n, p \in \mathbb{Z}\} = \{(a,\bar{b}) | (amp, \bar{b}mp + a\bar{y}) = (0,\bar{0}) \mid \text{for all } (mp,\bar{y}) \in S\} = \{(0, t\bar{\mu}) || \text{for all } t \in \mathbb{Z}\}$. So, $S(0,\bar{\mu}) = \{(x,\bar{y})(0,\bar{\mu}) | \text{for all } (x,\bar{y}) \in S\} = \{(0, x\bar{\mu}) | \text{for all } x \in \mathbb{Z}\}$. Therefore, $l_S(r_S((0,\bar{\mu}))) = S(0,\bar{\mu})$, for all $\bar{\mu} \in \mathbb{Z}_n$. Thus, S is right nil-injective ring.

Proposition 2.4 Let $S = \mathbb{Z} \oplus \mathbb{Z}_n$ and $\mathbb{Z}_n$ has non-zero nilpotent element. Then, S is right nil-injective if $r_{\mathbb{Z}_n}(\bar{p}) = \bar{p}\mathbb{Z}_n$, for each $\bar{p} \in N(\mathbb{Z}_n)$.

Proof. Suppose that $S = \mathbb{Z} \oplus \mathbb{Z}_n = \{(a,\bar{b}) | a \in \mathbb{Z} \text{ and } \bar{b} \in \mathbb{Z}_n\}$ is a ring with addition defined and multiplication defined by $(a,\bar{b})(c,\bar{d}) = (ac, \bar{b}\bar{d})$. It is clear that $N(S) = \{(0,\bar{p}) | \bar{p} \in N(\mathbb{Z}_n)\}$. We obtain that, $r_S((0,\bar{p})) = \{(x,\bar{y}) | \bar{p}\bar{y} = 0, x \in \mathbb{Z} \text{ and } \bar{y} \in \mathbb{Z}_n\} = \{(x,\bar{y}) | x \in \mathbb{Z} \text{ and } \bar{y} \in r_{\mathbb{Z}_n}(\bar{p})\}$. Since $r_{\mathbb{Z}_n}(\bar{p}) = \bar{p}\mathbb{Z}_n$, then $l_S\left(r_S((0,\bar{p}))\right) = \{(\alpha,\bar{\beta}) | (\alpha,\bar{\beta})(x,\bar{y}) = (0,\bar{0}), \text{for all } x \in \mathbb{Z} \text{ and } \bar{y} \in r_{\mathbb{Z}_n}(\bar{p})\} = \{(0,\bar{\beta}) | \bar{\beta} \in \bar{p}\mathbb{Z}_n\}$. So, $S(0,\bar{p}) = \{(x,\bar{y})(0,\bar{p}) | \text{ for all } (x,\bar{y}) \in S\} = \{(0,\bar{y}\bar{p}) | \text{ for all } \bar{y} \in \mathbb{Z}_n\} = \{(0,\bar{\beta}) | \bar{\beta} \in \bar{p}\mathbb{Z}_n\}$. Therefore, $l_S\left(r_S((0,\bar{p}))\right) = S(0,\bar{p})$ for each non-zero nilpotent element $\bar{p} \in N(\mathbb{Z}_n)$. Hence, S is right nil-injective ring.

Proposition 2.5 Let R be a local right nil-injective ring. Then for any non-zero (two-sided) ideals κR and νR of R, $\kappa R \cap \nu R \neq 0$, for any $\kappa, \nu \in N(R)$.

Proof. Suppose that $\kappa R \cap \nu R = 0$ and define the map $f: (\kappa+\nu)R \to R$ by $f[(\kappa+\nu)\chi] = \nu\chi$ for $k \in R$. Let $(\kappa+\nu)\chi = (\kappa+\nu)\chi'$ for $\chi, \chi' \in R$. So $\kappa(\chi - \chi') = \nu(\chi' - \chi) = 0$, yielding $\nu\chi' = \nu\chi$. Thus, f is well-defined. Since R is right nil-injective, then f can be extended on R. Therefore, $f[(\kappa+\nu)] = (\kappa+\nu)\omega$, for some $\omega \in R$. Thus, $b = (\kappa+\nu)\omega$. Since R local, then by [Proposition 7.2.11.,[6]] either ω or $1-\omega$ is a unit, but $0 = \kappa\omega = \nu(1-\omega) \in \kappa R \cap \nu R = \{0\}$. Thus, $\kappa = 0$ or $\nu = 0$, a contradiction. Hence, $\kappa R \cap \nu R \neq 0$, for any $\kappa, \nu \in N(R)$.

Proposition 2.6 Let R_R be a right nil-injective ring. Let $\kappa, \nu \in N(R)$:

(1) If $\kappa R \cong \nu R$ and an idempotent ϱ generates νR. Then there exists an idempotent $\vartheta \in R$ such that $\kappa = R\vartheta$, $r_R(\vartheta) = r_R(\kappa)$ and $R\kappa$ is a direct summand of R.

(2) If κR and νR are generated by two idempotent elements with $\kappa R \cap \nu R = 0$, then there exists an idempotent δ such that $\kappa R \oplus \nu R = \delta R$.

Proof. (1) Suppose that $\nu R = \varrho R$, for some $\varrho^2 = \varrho \in R$ and $\kappa R \cong \nu R$, we define $\sigma: \kappa R \to \nu R$ is an isomorphism, then $\sigma(\kappa) = \nu d$, for some $d \in R$ and $\sigma(\kappa c) = \varrho$, for some $c \in R$. Now, $\nu dc = \sigma(\kappa)c = \sigma(\kappa c) = \varrho$. Since $\nu R = \varrho R, \nu d = \varrho k$, for some $k \in R$. So, $\vartheta^2 = (c\nu d)(c\nu d) = c\varrho\nu d = c\varrho\varrho k = c\varrho k = c\nu d = \vartheta$. Thus, ϑ is an idempotent. So, $\kappa f = \kappa c\nu d = \sigma^{-1}(\varrho)b\nu d = \sigma^{-1}(\varrho b\nu d) = \sigma^{-1}(\varrho\varrho k) = \sigma^{-1}(\varrho k) = \sigma^{-1}(\nu d) = \kappa$. Let $x \in r_R(\vartheta)$, then $\kappa x = \kappa\vartheta x = 0$, so $r_R(\vartheta) \subseteq r_R(\kappa)$. But, as R is a right nil-injective. Then, ϑ is an idempotent and $R\kappa \subseteq R\vartheta$. Now, let $x \in r_R(\kappa)$, then $\vartheta x = c\nu dx = c\sigma(\kappa)x = c\sigma(\kappa x) = \sigma(0)c = 0$, so $r_R(\kappa) \subseteq r_R(\vartheta)$. But, as R is a right nil-injective, then $Rf \subseteq R\kappa$. Therefore, $Ra = Rf$ and $r_R(\kappa) = r_R(\vartheta)$. This gives $\vartheta = p\kappa$, for some $p \in R$. Since $\kappa = \kappa\vartheta$, we get $\kappa = \kappa p\kappa$ and so $R\kappa = R\vartheta = Rp\kappa = Rt$, where $t = p\kappa$ and $t^2 = (p\kappa)^2 = p\kappa p\kappa = p\kappa = t \in R$. Now, κR is a direct summand of R. We have to prove that $R = Rt \oplus R(1-t) = R\kappa \oplus R(1-t)$. Let $x \in Rt \cap R(1-t)$. Then, $x = rt \in tR$ and $r(1-t) \in R(1-t)$. Then, $2rt = r$. Thus, $2t = 1$. Since t is an idempotent, then $4t = 1$. Therefore, $4t - 2t = 1 - 1$. Then, $2t = 0$. Thus, $x = 0$. Hence, $R = Rt \oplus R(1-t) = R\kappa \oplus R(1-t)$.

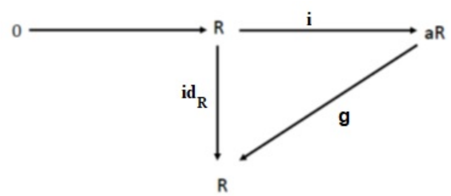
(2) Suppose that $aR = \varrho R$ and $bR = (1-\varrho)R$, for some idempotents ϱ and $(1-\varrho)$ of R. Then, $\kappa R \oplus \nu R = \varrho R \oplus \nu R = \varrho R \oplus (1-\varrho)R$ [as $\varrho, b \in (\varrho R \oplus (1-\varrho)R)$ and $\varrho, (1-\varrho) \in (aR \oplus bR)$]. Now, $\varrho R \oplus \nu R = \varrho R \oplus (1-\varrho)R$ implies $\nu R \cong (1-\varrho)R$. So, by (1) , $(1-\varrho)R = gR$, $g^2 = (1-\varrho)^2 = 1 - 2\varrho + \varrho^2 = 1 - 2\varrho + \varrho = 1 - \varrho = g \in R$ and $\varrho g = \varrho(1-\varrho) = 0$. Therefore, $\kappa R \oplus \nu R = \varrho R \oplus \nu R = \varrho R \oplus gR = (\varrho + g - \varrho g)R$ (since $\varrho + g - \varrho g = 1.\varrho + (1-\varrho)g \in (R\varrho \oplus Rg)$ and $\varrho = \varrho(\varrho + g - \varrho g) = \varrho + eg - eg \in R(\varrho + g - \varrho g)$. Therefore, $g = g(\varrho + g - \varrho g) = g\varrho + g - g\varrho g \in R(\varrho + g - \varrho g)$). Thus, $R\kappa \oplus R\nu = Rh$, where $h^2 = (\varrho + g - \varrho g)(\varrho + g - \varrho g) = (\varrho + g - \varrho g) = h \in R$. Hence, $R\varrho \oplus R\nu$ is a direct summand of R.

Theorem 2.7. Let R be a right Wnil- injective ring. If bR embeds in aR, where $r_R(b) = 0$, then there exists a positive integer number n such that $b^n R$ is an image of aR.

Proof. If $\sigma : bR \to aR$ is monic. Since R is a right Wnil-injective, there exists a positive integer n such that any right R-homomorphism of b^nR into R extends to one of R into R. Let right R-homomorphism $f = \iota\sigma i : b^nR \to R$, where $i : b^nR \to bR$ and $\iota : aR \to R$ are embedation maps. Hence $\sigma(b^n) = b^n v = ua$, where $v, u \in R$. Now let $\varphi : aR \to b^nR$, via: $\varphi(ar) = uar = b^n vr$. Since $b^n v \in N(R)$, there exists a positive integer m such that $(b^n v)^m R = r_R\,(l_R((b^n v)^m))$. Since $l_R((b^n v)^m) = l_R(b^n v) = l_R(b^n) = l_R(b) = 0, (b^n v)^m R = r_R\,(l_R((b^n v)^m)) = R$. Let $b^n = (b^n v)^m c$, where $c \in R$. Hence $\varphi(a(b^n v)^{m-1}\,c) = ua(b^n v)^{m-1}\,c = (b^n v)^m c = b^n$ and so φ is an epic.

Proposition 2.8. If R_R is a nil-injective ring, then aR is a direct summand of R, for all a ∈ N(R).

Proof. Let R_R be a nil-injective ring and consider the row exact diagram of R-modules,



Let id_R is the identity mapping on R and i is the canonical injection. If $g{:}\,aR \to R$ completes the diagram commutatively, then $gi = id_R$. Hence, g is a splitting map for i. If $v \in R$, then $g(v) \in R$, so $i(g(v)) \in aR$. If $\kappa = v - i(g(x))$, since $gi = id_R$. Then, $g(\kappa) = g(v) - g(i(g(v))) = 0$ Thus, $\kappa \in Kerg$ and $v = i(g(v)) + \kappa \in Imi + Kerg$. Therefore, $R = Imi + Kerg$. If $\lambda \in Imi \cap Kerg$, then $\lambda = i(x)$ for some $v \in R$, so $0 = g(y) = g(i(v)) = v$. Hence, $\lambda = 0$ and we have $R = Imi \oplus Kerg$. Since $Imi = aR$, then $R = aR \oplus Kerg$. Hence, aR is direct sumand of R.

Definition 2.9. A given R is a ring if it satisfies the following two conditions:

(1) For any right ideal χ, there is an idempotent ϱ such that ϱR is an essential extension of χ.

(2) If $\delta R, \delta = \delta^2$, is isomorphic to a right ideal Γ, then Γ also is generated by an idempotent.

A ring R is right continuous [12] if it satisfies Conditions 1 and 2.

Lemma 2.10. [Lemma4.1, [12]] If R is a right continuous ring, then $Z\ (R_R\ )\ =\ J(R)$, and $R/J(R)$ is regular.

Lemma 2.11. [Lemma 2.1,[11]] If Z (R_R) contains no non-zero nilpotent element, then Z(R_R)=0.

The following results are about the relation between right nil-injective ring and right continuous rings:

Proposition 2.12. Let R be a ring such that R is right continuous ring and $R_{R/J(R)}$ is nil injective ring. Then R is semiprimitive.

Proof. By Lemma 2.10**,** $J(R) = Z(R_R)$. We shall show that $J(R) = Z(R_R) = 0$. If not, by Lemma 2.11, there exists $0 \neq \kappa \in N(R)$ then $\kappa \in J(R)$. Since R a right continuous ring, then by Lemma 2.10, $R/J(R)$ is nil-injective, any R-homomorphism of κR into $R/J(R)$ extends to one of R into $R/J(R)$. Let $f{:}\,\kappa R \to R/J(R)$ such that $f(\kappa r) = r + J(R)$ where $r \in R$, we have to show that f is well defined, let $\kappa x = \kappa y,$ where $\alpha, \beta \in R$ then $\kappa(\alpha - \beta) = 0$. Thus, $(\alpha - \beta) + J(R) = J(R), \alpha + J(R) = \beta + J(R), f(x) = \alpha + J(R) = \beta + J(R) = f(\beta), f(\alpha) = f(\beta),$ so f is well defined right R-homomorphism, since $R/J(R)$ is nil-injective, there exists such that $1 + J(R) = f(\kappa) = (\nu + J(R))(\kappa + J(R)) = \nu\kappa + J(R)$, then $1 + J(R) = \nu\kappa + J(R)$. So $1 - \nu\kappa \in J(R)$. Since $\kappa \in J(R)$, then $1 - \nu\kappa$ is invertible. We get that $1 \in J(R)$, which is a contradiction. Therefore, $\kappa \notin J(R)$. So, $J(R) = 0$. This shows that R is semiprimitive.

We construct a relation between right Wnil-injective and right nil-injective in the matrix ring as follow:

Lemma 2.13. [Theorem2.3, [8]] A given ring R is right Wnil-injective if and only if for any $0 \neq \mathrm{a} \in \mathrm{N(R)}$, there exists a positive integer n such that $\kappa^{\mathrm{n}} \neq 0$ and $\mathrm{l_R(r_R(\kappa^n))} = \mathrm{R}\kappa^{\mathrm{n}}$.

Theorem 2.14. Let R be a ring and $\mathrm{S} = \mathrm{M_n(R)}$ be the matrix ring. Let $\kappa E_{n1} = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \kappa & 0 & . & . & . & 0 \end{pmatrix}$, for $\kappa \in \mathrm{N(R)}$, then the followings are true:

(1) $\mathrm{l_S(r_S(\kappa E_{n1}))} = \mathrm{S\kappa E_{n1}}$ if and only if $\mathrm{l_R(r_R(\kappa))} = \mathrm{R\kappa}$.

(2) If $M_n(R)$ is a right Wnil-injective ring, for some $n \geq 2$, then R is a right nil-injective ring.

Proof. 1. Let $b \in l_R(r_R(\kappa))$ then $r_R(\kappa) \subseteq r_R(\nu)$. Now, take $(\omega_{ij}) \in r_S(\kappa E_{n1})$, then

$$,\begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \kappa & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} \omega_{11} & \omega_{12} & . & . & . & \omega_{1n} \\ \omega_{21} & \omega_{22} & . & . & . & \omega_{2n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \omega_{n1} & \omega_{n2} & . & . & . & \omega_{nn} \end{pmatrix} = 0$$

so we have $\kappa\omega_{1i} = 0$, for all $i = 1,2,\ldots,n$. That is, $\omega_{1i} \in r_R(\kappa) \subseteq r_R(\nu)$ so $\nu\omega_{1i} = 0$, for $i = 1,2,\ldots n$, yielding $(\nu E_{n1})(\omega_{1i}) = 0$. Thus, $(\omega_{ij}) \in r_S(\nu E_{n1})$, hence $r_S(E_{n1}\kappa) \subseteq r_S(E_{n1}\nu)$. Therefore, $\nu E_{n1} \in l_S(r_S(\kappa E_{n1}) = S(\kappa E_{n1})$. So, we can write $\nu E_{n1} = (d_{ij})\kappa E_{n1}$, where $(d_{ij}) \in S$, which implies $\nu = d_{nn}\kappa \in R\kappa$. Hence, $l_R(r_R(\kappa)) = R\kappa$. Conversely, Let $B = (bij) \in l_S(r_S(\kappa E_{n1}))$ then $r_R(\kappa E_{n1}) \subseteq r_R(B)$. Now, if $i \neq 1$, then $(\kappa E_{n1})E_{ij} = 0$ which implies $E_{ij} \in r_S(\kappa E_{n1}) \subseteq r_S(B)$ thus $BE_{ij} = 0$ that is $(\nu_{ij})(E_{ij}) = 0$ hence $\nu_{ki} = 0$ for $k = 1,2,..,n$. So, $B = \begin{pmatrix} \nu_{11} & 0 & . & . & . & 0 \\ \nu_{21} & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \nu_{n1} & 0 & . & . & . & 0 \end{pmatrix}$. Then, If $\nu \in r_R(\kappa)$ then $\nu E_{11} \in r_S(\kappa E_{n1}) \subseteq r_S(B)$. So, $\nu \in r_R(\nu_{i1})$, for $i = 1,2,..,n$. Thus, $r_R(\kappa) \subseteq r_R(\nu_{i1})$ implies $l_R(r_R(\nu_{i1})) \subseteq l_R(r_R(\kappa))$ then $\nu_{i1} \in l_R(r_R(\nu_{i1})) \subseteq l_R(r_R(\kappa)) = R\kappa$. So, $\nu_{i1} = t_{i1}\kappa$ with $t_{i1} \in R$ for $i = 1,\ldots,n$. Thus $B = \begin{pmatrix} t_{11}\kappa & 0 & . & . & . & 0 \\ t_{21}\kappa & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ t_{n1}\kappa & 0 & . & . & . & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & . & . & . & t_{11} \\ 0 & 0 & . & . & . & t_{21} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & & & & . & . \\ 0 & 0 & . & . & . & t_{n1} \end{pmatrix}(\kappa E_{n1}) \in S(\kappa E_{n1})$. Therefore, $l_S(r_S(\kappa E_{n1})) = S(\kappa E_{n1})$.

(2) Let $0 \neq \kappa \in N(R)$ and take, $u = \kappa E_{n1}$. Now, $M_n(R)$ is right Wnil-injective. So, by **Lemma 2.13.** there exists $m > 1$ such that $u^m \neq 0$ and $l_S(r_S(u)) = Su^m$. Since $n \geq 2$, $u^2 =$

$$\begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \kappa & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \kappa & 0 & . & . & . & 0 \end{pmatrix} = 0.$$ So it must be that $m = 1$ and $l_S(r_S(u)) = Su$. Thus R is right nil-injective.

A non-zero right R-module M is said to be s-unital [4], if $u \in uR$ for each $u \in M$. If R_R is s-unital, then R is called a right s-unital ring. If M is a right R-module and S is a subset of R, then we set $l_M(S) = \{u \in M | uS = 0\}$. HIRANO and TOMINAGA introduced in [13], if M is a right R-module and S is a subset of R, then $r_M(S) = \{u \in M | Su = 0\}$. So, if M is a right R-module and α is an element of R, then $r_M(\alpha) = \{u \in M | \alpha u = 0\}$. Finally, if M is a right R-module and α is an element of R, then $l_M(\alpha) = \{u \in M | u\alpha = 0\}$.

Theorem 2.15. [Theorem1, [4]] If F is a finite subset of a right s-unital ring R, then there exists an element $e \in R$ such that $\alpha e = \alpha$, for all $\alpha \in F$.

An R-module M is called right nil-injective module if each $a \in N(R)$ and each homomorphism $f: aR \to M$, there exists a homomorphism $g: R \to M$ such that $f(x) = g(x)$, for every $x \in \kappa R$ [2].

Theorem 2.16 Let M be s-unital module, then the following conditions are equivalent:

(1) M_R is a right nil-injective module.

(2) $l_M(r_R(\alpha)) = M\alpha$ for every $\alpha \in N(R)$.

(3) $r_R(\alpha) \subseteq r_R(\beta)$ where $\alpha, \beta \in N(R)$, then $\beta M \subseteq \alpha M$.

(4) If $f: \alpha R \to R, \alpha \in N(R)$, is R-linear, then $f(\alpha) \in M\alpha$.

Proof. (1) ⇒ (2) Assume that M_R is nil-injective. Given $u \in l_M(r_R(\alpha))$ such that $\alpha \in N(R)$ there exist an element $e' \in R$ such that $ue' = u$. Then, by **Theorem 2.15.**, there exists an element $e \in R$ such that $\alpha e = \alpha$ and $e'e = e'$. Consider $f: \alpha R \to M$ defined by $f(\alpha x) = ux$. Since M is a nil-injective, we can find an element $v \in M$ with $ux = v\alpha x$, for all $x \in R$. We therefore obtain $u =$

$ue' = ue = v\alpha e = v\alpha$, which means $l_M(r_R(\alpha)) \subseteq M\alpha$. On the other hand, let $va \in M\alpha$, for some $v \subseteq M$. Then, $v\alpha x = 0$, for every $x \in r_R(\alpha)$. Thus, $v\alpha \in l_M(r_R(\alpha))$. So that $l_M(r_R(\alpha)) = M\alpha$.

(2) ⇒ (3) Let $S_1 \subseteq S_2$. Then $l_M(S_2) = \{u \in M |\ uS_2 = 0\} \subseteq \{u \in M |\ uS_1 = 0 = l_M(S_2)\}$. Suppose $\alpha, \beta \in N(R)$ such that $r_R(\alpha) \subseteq r_R(\beta)$. Then, $l_M(r_R(\beta)) \subseteq l_M(r_R(\alpha))$. Therefore, $Mb = l_M(r_R(\alpha)) \subseteq l_M(r_R(\alpha)) = M\alpha$.

(3) ⇒ (4) First, $l_R(\beta) + R\alpha \subseteq l_R(\beta R \cap r_R(\alpha))$ as $x \in l_R(\beta) + R\alpha$ implies that $x = y + k\alpha$ where $y\beta = 0$. Now, we must show $x \in l_R[\beta R \cap r_R(\alpha)]$. Then, $x(\beta R \cap r_R(\alpha)) = 0$. Therefore, $(y + k\alpha)(\beta R \cap r_R(\alpha)) = 0$. We have, $(y + k\alpha)(\beta R \cap r_R(\alpha)) = \{(y + k\alpha)\beta bt | \alpha\beta t = 0, t \in R\} = \{y\beta t | t \in R = \{0\}\}$. Let $t = 1$, then $y\beta t = y\beta = 0$. Thus, $y \in l_R(\beta)$. Therefore $l_R(\beta) + R\alpha \subseteq l_R \beta R \cap r_R(\alpha))$. Now, let $x \in l_R(\beta R \cap r_R(\alpha))$, then $x(\beta R \cap r_R(\alpha)) = 0$. This means that $\{x\beta t | \alpha\beta t = 0, t \in R = 0\}$. So, whenever $t \in l_R(\alpha\beta), t \in l_R(x\beta)$ showing that $r_R(\alpha\beta) \subseteq r_R(x\beta)$ and so $Rx\beta \subseteq R\alpha\beta(by(3))$. This implies that $xb = p\alpha\beta$ for some $p \in R$ yielding $x - p\alpha \in l_R(\beta)$ that is $x \in l_R(\beta) + R\alpha$. Thus, $l_R(\beta R \cap r_R(\alpha)) \subseteq (l_R(\beta) + R\alpha)$. Hence, $l_R(\beta R \cap r_R(\alpha)) = l_R(\beta) + R\alpha$.

(4) ⇒ (1) Let $f: \alpha R \to M$ be R-linear map with $f(\alpha) \in M\alpha$. Then, $f(\alpha) = c\alpha$, for some $c \in M$. This proves (1). Which completes the proof.

3. Conclusions

In conclusion, our study has demonstrated examples of rings that are nil-injective but not p-injective. We also attempted to find examples of rings that are Wnil-injective but not nil-injective. These examples highlight the importance of studying these generalizations, as they different from the previous types of nil-injective rings.

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على بعض النتائج لحلقات من النمط nil

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المستخلص

يقال للمقاس الأيمن M انه غامر من النمط nil إذا كان لكل $\omega \in \mathrm{N(R)}$ وان أي هومومورفيزم $f: \omega R \to M$ الى توسعته يمكن $R \to M$. يقال للمقاس الأيمن R_R انه غامر من النمطnil إذا كان R غامر من النمط nil. وأيضا , يقال للمقاس الأيمن M انه غامر من النمطWnil إذا كان لكل $0 \neq \omega \in \mathrm{N(R)}$ يوجد عدد صحيح موجب n بحيث أن $\omega^n \neq 0$ وان أي هومومورفيزم $f: \omega^n R \to M$ الى توسعته يمكن $R \to M$. يقال للمقاس الأيمن R_R انه غامر من النمط$Wnil$ إذا كان R غامر من النمط $Wnil$. نناقش بعض الخصائص والتوصيفات المتعلقة بهذه الحلقات في هذا العمل.