

Continuous dependence of double diffusive convection in a porous medium with temperature-dependent density

Ayat A. Hameed and Akil J. Harfash
*Department of Mathematics, College of Sciences,
University of Basrah, Basrah, Iraq*

Doi 10.29072/basjs.20190101, Article inf., Received: 15/1/2019 Accepted: 3/4/2019 Published: 30/4/2019

Abstract

The structural stability of a double diffusive convection in a porous medium of the Forchheimer type was studied, when the density of fluid depends on temperature and concentration as a cubic and linear function, respectively. It has been shown that for this problem, with thermal convection in a plane infinite layer, the resonance can occur between the internal layers that arise. The main parameter is the internal heat source and its presence may lead to oscillatory convection in linear instability inducing resonance. Thus, in this study, the structural stability problem of continuous dependence on the heat source itself for a model of nonisothermal flow in a porous medium of Forchheimer type was analyzed. Furthermore, the continuous dependence of the solution on changes in the Forchheimer coefficients has been shown.

Keywords: Structural stability, Double diffusive, Darcy's law, Forchheimer theory, Cubic density.

1. Introduction

The problem of double diffusive convection in a horizontal layer of porous material saturated with an incompressible fluid has attracted the attention of many writers, see cf. Straughan [1]. An important category of such problem is the structural stability in porous media, or continuous dependence on the model itself. In general, in the field of continuum mechanics, or in partial differential equations, structural stability is prominent, cf. Hirsch and Smale [2]. The continuous dependence on modelling, for the elasticity field, was initiated in a seminal paper of Knops and Payne [3], and these authors have produced improved results in Knops and Payne [4]. Payne [5-7] also developed the field of structural stability, and since then many papers have emerged. References to these can be found in the field of porous media in chapter 2 of the book by Straughan [1], with recent contributions from Aulisa et al. [8], Ciarletta et al. [9], Hoang and Ibragimov [10], Harfash [11-13], Liu [14], Liu et al. [15, 16].

In this paper, we continue with Straughan [17] and Gentile and Straughan [18] who studied the continuous dependence on the heat source in a penetrative convection model in a Forchheimer porous medium when the density depends in quadratic and cubic manner on the temperature field, respectively. For many applications, the quadratic dependence is insufficient and a cubic dependence is necessary, cf. McKay and Straughan [19], Straughan ([20], pp. 143-144). Moreover, in a situation where fluid flow is not small, it is possible to introduce Forchheimer coefficients in the Darcy equations (see [21, 22]), with the idea being that the pressure gradient is no longer proportional to the velocity itself, cf. Straughan ([1], p. 12), Néel [23]. Here we deal with the Forchheimer model with quadratic degree. We first analyse the continuous dependence of the solution on changes in the heat source. Then we check the continuous dependence on Forchheimer coefficients. A separate analysis is provided for each of the parameters, which is necessary

because the bounds obtained are different in each case.

For Forchheimer theory, Straughan [24] showed that with heat source and nonlinear density, penetrative convection may occur simultaneously in different layers in a porous medium, resulting a resonance phenomenon. The oscillatory convection results from an interaction between the effects of nonlinear density and heat source. It is therefore important to demonstrate the continuous dependence on the heat source. Since the model we are studying is highly nonlinear, the analysis, as shown here, is non trivial.

2. Basic Equations

We take the momentum equation in a saturated material of Forchheimer type to have the form,

$$v_i + a|\mathbf{v}|v_i + b|\mathbf{v}|^2v_i = -\pi_{,i} + g_iT + h_iT^2 + L_iT^3 + I_iC, \quad (1)$$

where v_i , T and π are velocity, temperature and pressure, a and b are Forchheimer coefficients, and g_i, h_i, L_i and I_i are vectors incorporating the gravity field which, without loss of generality, we take such that $|\mathbf{g}| \leq 1$, $|\mathbf{h}| \leq 1$, $|\mathbf{L}| \leq 1$ and $|\mathbf{I}| \leq 1$. The standard indicial notation is assumed throughout, with, for example, subscript, i denoting $\partial/\partial x_i$, and subscript, t denoting $\partial/\partial t$. The balance of mass equation for an incompressible fluid is

$$v_{i,i} = 0, \quad (2)$$

while the temperature and concentration equations have, respectively, the following forms

$$T_i + v_iT_{,i} = \Delta T + Q, \quad (3)$$

$$C_{,t} + v_iC_{,i} = \Delta C. \quad (4)$$

Let Ω be a bounded domain in \mathbb{R}^3 with boundary Γ smooth enough to allow application of the divergence theorem. Then, Equations (1)-(3) are defined on $\Omega \times (0, \mathcal{T}]$, for $\mathcal{T} < \infty$ a fixed time. The boundary conditions we employ are that

$$v_i n_i = 0, \quad \text{on} \quad \Gamma \times [0, \mathcal{T}], \quad (5)$$

and

$$T(\mathbf{x}, t) = h(\mathbf{x}, t), \quad C(\mathbf{x}, t) = k(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma, \quad t \in [0, \mathcal{T}], \quad (6)$$

where h and k are prescribed functions and n is the unit outward normal to Γ . The initial condition is

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}) \quad \text{and} \quad C(x, 0) = C_0(\mathbf{x}), \quad (7)$$

where T_0 and C_0 are prescribed functions. Let the boundary-initial value problem comprised of Equations (1)-(3) together with conditions (5)-(7) be denoted by \mathcal{P} .

3. A priori estimates

Firstly, we need to find some a priori estimates for various norms of T and C which are important to derive the continuous dependence of a solution of problem \mathcal{P} on the heat source Q and Forchheimer coefficients a and b . To find these estimates, the functions $G(\mathbf{x}, t), K(\mathbf{x}, t), I(\mathbf{x}, t), F(\mathbf{x}, t)$ and $H(\mathbf{x}, t)$ have been introduced as solutions to the following boundary value problems

$$\begin{aligned} \Delta G &= 0, & \text{in } \Omega, \\ G &= h(\mathbf{x}, t), & \text{on } \Gamma, \end{aligned} \quad (8)$$

$$\begin{aligned} \Delta K &= 0, & \text{in } \Omega, \\ K &= k(\mathbf{x}, t), & \text{on } \Gamma, \end{aligned} \quad (9)$$

$$\begin{aligned} \Delta I &= 0, & \text{in } \Omega, \\ I &= h^3(\mathbf{x}, t), & \text{on } \Gamma, \end{aligned} \tag{10}$$

$$\begin{aligned} \Delta F &= 0, & \text{in } \Omega, \\ F &= h^5(\mathbf{x}, t), & \text{on } \Gamma, \end{aligned} \tag{11}$$

and then last step is

$$\begin{aligned} \Delta H &= 0, & \text{on } \Omega, \\ H &= h^{2p-1}(\mathbf{x}, t), & \text{in } \Gamma, \end{aligned} \tag{12}$$

where p is a positive integer to be specified later. Now, multiply Equation (1) by v_i integrate over Ω , and apply the Cauchy-Schwarz and arithmetic-geometric mean inequalities, to see that

$$\begin{aligned} \|\mathbf{v}\|^2 + a \|\mathbf{v}\|_3^3 + b \|\mathbf{v}\|_4^4 &= (g_i T, v_i) + (h_i T^2, v_i) + (L_i T^3, v_i) + (I_i C, v_i) \\ &\leq (T, v_i) + (T^2, v_i) + (T^3, v_i) + (C, v_i) \\ &\leq \frac{1}{6} \|\mathbf{v}\|^2 + \frac{3}{2} \|T\|^2 + \frac{1}{6} \|v\|^2 + \frac{3}{2} \|T\|_4^4 \\ &\quad + \frac{1}{6} \|\mathbf{v}\|^2 + \frac{3}{2} \|T\|_6^6 + \frac{1}{6} \|v\|^2 + \frac{3}{2} \|C\|^2 \\ &= \frac{2}{3} \|\mathbf{v}\|^2 + \frac{3}{2} (\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2), \end{aligned} \tag{13}$$

Thus, from inequality (13), it follows that

$$\frac{1}{3} \|\mathbf{v}\|^2 + a \|\mathbf{v}\|_3^3 + b \|\mathbf{v}\|_4^4 \leq \frac{3}{2} (\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2), \tag{14}$$

which leads to the following bounds

$$\begin{aligned} \|\mathbf{v}\|^2 &\leq \frac{9}{2} (\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2), \\ \|\mathbf{v}\|_3^3 &\leq \frac{3}{2a} (\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2), \\ \|\mathbf{v}\|_4^4 &\leq \frac{3}{2b} (\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2). \end{aligned} \tag{15}$$

We now form the expressions

$$\int_0^t \int_{\Omega} (T - G)(T_{,s} + v_i T_{,i} - \Delta T - Q) dx ds = 0, \tag{16}$$

$$\int_0^t \int_{\Omega} (C - K)(C_{,s} + v_i C_{,i} - \Delta C) dx ds = 0, \tag{17}$$

$$\int_0^t \int_{\Omega} (T^3 - I)(T_{,s} + v_i T_{,i} - \Delta T - Q) dx ds = 0, \tag{18}$$

and

$$\int_0^t \int_{\Omega} (T^5 - F)(T_{,s} + v_i T_{,i} - \Delta T - Q) dx ds = 0, \tag{19}$$

where t is some number such that $0 < t \leq T$. Next, integrate by parts in (16) and employ the boundary condition (8)₂ to see that

$$\begin{aligned} \frac{1}{2} \|T\|^2 + \int_0^t \|\nabla T\|^2 ds &\leq \frac{1}{2} \|T_0\|^2 + \int_0^t (T, Q) ds + (T, G) + |(T_0, G_0)| \\ &\quad + |\int_0^t (G_{,s}, T) ds| + \int_0^t \int_{\Omega} G v_i T_{,i} dx ds + \int_0^t \oint_{\Gamma} h \left(\frac{\partial G}{\partial n}\right) dA ds + |\int_0^t (G, Q) ds|. \end{aligned} \tag{20}$$

We next bound the cubic term on the right of (20) as follows, where G_m is the maximum of G on $\Gamma \times [0, T]$,

$$\begin{aligned}
 \int_0^t \int_{\Omega} G v_i T_{,i} d\mathbf{x} ds &\leq G_m \sqrt{\int_0^t \| \mathbf{v} \|^2 ds \int_0^t \| \nabla T \|^2 ds} \\
 &\leq \frac{3G_m}{\sqrt{2}} \sqrt{\int_0^t (\| T \|^2 + \| C \|^2 + \| T \|^4 + \| T \|^6) ds} \int_0^t \| \nabla T \|^2 ds \\
 &\leq \frac{9G_m^2 \alpha_1}{4} \int_0^t (\| T \|^2 + \| C \|^2 + \| T \|^4 + \| T \|^6) ds + \frac{1}{2\alpha_1} \int_0^t \| \nabla T \|^2 ds,
 \end{aligned} \tag{21}$$

where we have used the Cauchy-Schwarz and arithmetic-geometric mean inequalities and where $\alpha_1 > 0$ is a constant to be chosen. Then, by using the Cauchy-Schwarz and arithmetic-geometric mean inequalities and with $\alpha_1 = 2$, we arrive at

$$\begin{aligned}
 \frac{1}{2} \| T \|^2 + \frac{3}{4} \int_0^t \| \nabla T \|^2 ds &\leq 2 \| G \|^2 + \frac{1}{8} \| T \|^2 + \| T_0 \|^2 + \frac{1}{2} \| G_0 \|^2 + \frac{1}{2} \int_0^t \| G_{,s} \|^2 ds \\
 + (1 + \frac{9G_m}{2}) \int_0^t \| T \|^2 ds + \frac{9G_m^2}{2} \int_0^t (\| C \|^2 + \| T \|^4 + \| T \|^6) ds \\
 + \frac{1}{2} \int_0^t \oint_{\Gamma} h^2 dA ds + \frac{1}{2} \int_0^t \oint_{\Gamma} (\frac{\partial G}{\partial n})^2 dA ds + \frac{1}{2} \int_0^t \| G \|^2 ds + \int_0^t \| Q \|^2 ds.
 \end{aligned} \tag{22}$$

We now return to Equation (17) and perform various integrations by parts to find with the aid of the Cauchy-Schwarz and arithmetic-geometric mean inequalities,

$$\begin{aligned}
 \frac{1}{2} \| C \|^2 + \int_0^t \| \nabla C \|^2 ds &\leq \frac{4}{3} \| K \|^2 + \frac{3}{16} \| C \|^2 + \| C_0 \|^2 + \frac{1}{2} \| K_0 \|^2 \\
 + \frac{1}{2\alpha_2} \int_0^t \| K_{,s} \|^2 ds + \frac{\alpha_2}{2} \int_0^t \| C \|^2 ds + \frac{1}{4} \int_0^t \| \nabla C \|^2 ds \\
 + K_m^2 \int_0^t \| \mathbf{v} \|^2 ds + \frac{1}{2} \int_0^t \oint_{\Gamma} k^2 dA ds + \frac{1}{2} \int_0^t \oint_{\Gamma} (\frac{\partial K}{\partial n})^2 dA ds,
 \end{aligned} \tag{23}$$

where K_m is the maximum value of K on $\Gamma \times [0, T]$. Inequality (15)₁ is next employed in (23) and then with further use of the arithmetic-geometric mean inequality in (23) we may show that

$$\begin{aligned}
 \frac{5}{16} \| C \|^2 + \frac{3}{4} \int_0^t \| \nabla C \|^2 ds &\leq \frac{4}{3} \| K \|^2 + \| C_0 \|^2 + \frac{1}{2} \| K_0 \|^2 + \frac{1}{2} \int_0^t \| K_{,s} \|^2 ds \\
 + (\frac{1}{2} + \frac{9K_m^2}{2}) \int_0^t \| C \|^2 ds + \frac{9K_m^2}{2} \int_0^t (\| T \|^2 + \| T \|^4 + \| T \|^6) ds \\
 + \frac{1}{2} \int_0^t \oint_{\Gamma} k^2 dA ds + \frac{1}{2} \int_0^t \oint_{\Gamma} (\frac{\partial K}{\partial n})^2 dA ds.
 \end{aligned} \tag{24}$$

Next, after integration by parts in (18) and some rearrangement we may produce

$$\begin{aligned}
 \frac{1}{4} \| T \|^4 - \frac{1}{4} \| T_0 \|^4 + \frac{3}{4} \int_0^t \| \nabla T^2 \|^2 ds - \int_0^t (T^3, Q) ds - (I, T) + (I_0, T_0) \\
 + \int_0^t (T, I_{,s}) ds - \int_0^t \int_{\Omega} I v_i T_{,i} d\mathbf{x} ds + \int_0^t (I, Q) ds - \int_0^t \oint_{\Gamma} h (\frac{\partial I}{\partial n}) dA ds = 0.
 \end{aligned} \tag{25}$$

Hence, with further use of the Cauchy-Schwarz and the arithmetic-geometric mean inequalities, from (25) we deduce

$$\begin{aligned}
 \frac{1}{4} \| T \|^4 + \frac{3}{4} \int_0^t \| \nabla T^2 \|^2 ds &\leq \frac{1}{2} \int_0^t \| I_{,s} \|^2 ds + (\frac{1}{2} + \frac{9I_m^2}{2}) \int_0^t \| T \|^2 ds \\
 + (\frac{3}{4} + \frac{9I_m^2}{2}) \int_0^t \| T \|^4 ds + \frac{1}{4} \| T_0 \|^4 + 4 \| I \|^2 + \frac{1}{2} \| I_0 \|^2 + \frac{1}{2} \| T_0 \|^2 \\
 + \frac{1}{4} \int_0^t \| \nabla T \|^2 ds + \frac{9I_m^2}{2} \int_0^t (\| C \|^2 + \| T \|^6) ds + \frac{1}{2} \int_0^t \oint_{\Gamma} h^2 dA ds \\
 + \frac{1}{2} \int_0^t \oint_{\Gamma} (\frac{\partial I}{\partial n})^2 dA ds + \frac{1}{2} \int_0^t \| I \|^2 ds + \frac{1}{2} \int_0^t \| Q \|^2 ds + \frac{1}{4} \int_0^t \| Q \|^4 ds.
 \end{aligned} \tag{26}$$

Form the identity (19), by integrations by parts one then finds

$$\begin{aligned}
 \frac{1}{6} \| T \|^6 - \frac{1}{6} \| T_0 \|^6 + \frac{5}{9} \int_0^t \| \nabla T^3 \|^2 ds - \int_0^t \oint_{\Gamma} h^5 (\frac{\partial T}{\partial n}) dA ds - \int_0^t (T^5, Q) ds \\
 + (F_0, T_0) + \int_0^t (F_{,s}, T) ds - \int_0^t \int_{\Omega} F v_i T_{,i} d\mathbf{x} ds + \int_0^t \oint_{\Gamma} h^5 (\frac{\partial T}{\partial n}) dA ds \\
 + \int_0^t (F, Q) ds - \int_0^t \oint_{\Gamma} h (\frac{\partial F}{\partial n}) dA ds = 0.
 \end{aligned} \tag{27}$$

Next, making further use of the arithmetic-geometric mean inequality in (27) we may derive

$$\begin{aligned} & \frac{1}{6} \| T \|^6 + \frac{5}{9} \int_0^t \| \nabla T^3 \|^2 ds \leq \frac{1}{6} \| T_0 \|^6 + \frac{\beta}{2} \| F \|^2 + \frac{1}{2\beta} \| T \|^2 + \frac{5}{6} \int_0^t \| T \|^6 ds \\ & + \frac{1}{6} \int_0^t \| Q \|^6 ds + \frac{1}{2} \| F_0 \|^2 + \frac{1}{2} \| T_0 \|^2 + \frac{1}{2} \int_0^t \| F_{,s} \|^2 ds + \frac{1}{2} \int_0^t \| T \|^2 ds \\ & + \frac{9F_m^2}{2} \int_0^t (\| T \|^2 + \| C \|^2 + \| T \|^4 + \| T \|^6) ds + \frac{1}{4} \int_0^t \| \nabla T \|^2 ds \\ & + \frac{1}{2} \int_0^t \oint_{\Gamma} h^2 dAds + \frac{1}{2} \int_0^t \oint_{\Gamma} \left(\frac{\partial F}{\partial n}\right)^2 dAds + \frac{1}{2} \int_0^t \| F \|^2 ds + \frac{1}{2} \int_0^t \| Q \|^2 ds. \end{aligned} \tag{28}$$

Next, add (22), (24), (26) and (28) to obtain

$$\begin{aligned} & \frac{1}{4} \| T \|^2 + \frac{5}{16} \| C \|^2 + \frac{1}{4} \| T \|^4 + \frac{1}{6} \| T \|^6 + \frac{1}{4} \int_0^t \| \nabla T \|^2 ds \\ & + \frac{3}{4} \int_0^t \| \nabla C \|^2 ds + \frac{3}{4} \int_0^t \| \nabla T^2 \|^2 ds + \frac{5}{9} \int_0^t \| \nabla T^3 \|^2 ds \\ & \leq \left[2 + \frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2} \right] \int_0^t \| T \|^2 ds \\ & + \left[\frac{9G_m^2}{2} + \frac{1}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2} \right] \int_0^t \| C \|^2 ds \\ & + \left[\frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{3}{4} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2} \right] \int_0^t \| T \|^4 ds \\ & + \left[\frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2} + \frac{5}{6} \right] \int_0^t \| T \|^6 ds + E(t), \end{aligned} \tag{29}$$

where $E(t)$ is a term we will show is bounded by data and is defined by

$$\begin{aligned} E(t) = & 2 \| G \|^2 + \frac{4}{3} \| K \|^2 + 4 \| F \|^2 + 4 \| I \|^2 + 2 \| T_0 \|^2 + 2 \int_0^t \| Q \|^2 ds + \frac{1}{2} \| G_0 \|^2 \\ & + \frac{1}{2} \| K_0 \|^2 + \frac{1}{2} \| I_0 \|^2 + \frac{1}{2} \| F_0 \|^2 + \frac{1}{6} \| T_0 \|^6 + \| C_0 \|^2 + \frac{1}{4} \| T_0 \|^4 \\ & + \frac{1}{2} \int_0^t \| G_{,s} \|^2 ds + \frac{1}{2} \int_0^t \| K_{,s} \|^2 ds + \frac{1}{2} \int_0^t \| I_{,s} \|^2 ds + \frac{1}{2} \int_0^t \| F_{,s} \|^2 ds \\ & + \frac{1}{2} \int_0^t \oint_{\Gamma} k^2 dAds + \frac{3}{2} \int_0^t \oint_{\Gamma} h^2 dAds + \frac{1}{2} \int_0^t \oint_{\Gamma} \left(\frac{\partial G}{\partial n}\right)^2 dAds \\ & + \frac{1}{2} \int_0^t \oint_{\Gamma} \left(\frac{\partial K}{\partial n}\right)^2 dAds + \frac{1}{2} \int_0^t \oint_{\Gamma} \left(\frac{\partial I}{\partial n}\right)^2 dAds + \frac{1}{2} \int_0^t \oint_{\Gamma} \left(\frac{\partial F}{\partial n}\right)^2 dAds \\ & + \frac{1}{2} \int_0^t \| G \|^2 ds + \frac{1}{2} \int_0^t \| I \|^2 ds + \frac{1}{2} \int_0^t \| F \|^2 ds + \frac{1}{6} \int_0^t \| Q \|^6 ds + \frac{1}{4} \int_0^t \| Q \|^4 ds. \end{aligned} \tag{30}$$

Payne and Straughan [25] show that for a function ϕ satisfying

$$\begin{aligned} \Delta \phi &= 0, \quad \text{in } \Omega, \\ \phi &= M, \quad \text{on } \Gamma, \end{aligned} \tag{31}$$

then one may use a Rellich identity, cf. Payne and Weinberger [26], to determine constants c_1, c_2 such that

$$\| \nabla \phi \|^2 + c_1 \oint_{\Gamma} \left(\frac{\partial \phi}{\partial n}\right)^2 dA \leq \oint_{\Gamma} c_2 |\nabla_s M|^2 dA, \tag{32}$$

where ∇_s denotes the surface gradient over the boundary. They also show that

$$2(\psi \nabla \phi, \nabla \phi) + \| \phi \|^2 \leq \psi_1 \oint_{\Gamma} M^2 dA, \tag{33}$$

where

$$\psi_1 = \max_{\Gamma} \left| \frac{\partial \psi}{\partial n} \right|,$$

with ψ solving the boundary value problem,

$$\begin{aligned} \Delta \psi &= -1, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \Gamma. \end{aligned}$$

Thus, inequalities (33) and (32) lead to bounds for $E(t)$ in terms of data. In fact, one may show

$$E(t) \leq D(t), \tag{34}$$

where

$$\begin{aligned} D(t) = & 2 \| T_0 \|^2 + \frac{1}{4} \| T_0 \|^4 + \frac{1}{6} \| T_0 \|^6 + \| K_0 \|^2 + 2 \int_0^t \| Q \|^2 ds \\ & + \frac{1}{4} \int_0^t \| Q \|^4 ds + \frac{1}{6} \int_0^t \| Q \|^6 ds + (\frac{3}{2} + \psi_1) \int_0^t \oint_{\Gamma} h^2 dAds \\ & + \frac{1}{2} \int_0^t \oint_{\Gamma} k^2 dAds + 2\psi_1 \oint_{\Gamma} h^2 dA + \frac{\psi_1}{2} \oint_{\Gamma} h_0^2 dA + 4\psi_1 \oint_{\Gamma} h^6 dA \\ & + \frac{\psi_1}{2} \oint_{\Gamma} h_0^6 dA + 4\psi_1 \oint_{\Gamma} h^{10} dA + \frac{\psi_1}{2} \oint_{\Gamma} h_0^{10} dA + \frac{4}{3} \psi_1 \oint_{\Gamma} k^2 dA + \frac{\psi_1}{2} \oint_{\Gamma} h_0^2 dA \\ & + \frac{c_2}{2c_1} \int_0^t \oint_{\Gamma} |\nabla_s h|^2 dAds + \frac{c_2}{2c_1} \int_0^t \oint_{\Gamma} |\nabla_s h^3|^2 dAds + \frac{c_2}{2c_1} \int_0^t \oint_{\Gamma} |\nabla_s h^5|^2 dAds \\ & + \frac{c_2}{2c_1} \int_0^t \oint_{\Gamma} |\nabla_s k|^2 dAds + \frac{\psi_1}{2} \int_0^t \oint_{\Gamma} h^4 h_{,s}^2 dAds + \frac{\psi_1}{2} \int_0^t \oint_{\Gamma} h^8 h_{,s}^2 dAds \\ & + \frac{\psi_1}{2} \int_0^t \oint_{\Gamma} k_{,s}^2 dAds + \frac{\psi_1}{2} \int_0^t \oint_{\Gamma} h^6 dAds + \frac{\psi_1}{2} \int_0^t \oint_{\Gamma} h^{10} dAds + \| C_0 \|^2. \end{aligned} \tag{35}$$

We put

$$\begin{aligned} \mathcal{K} = & \max\{4(2 + \frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2}), \frac{16}{5}(\frac{9G_m^2}{2} + \frac{1}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2}), \\ & 4(\frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{3}{4} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2}), 6(\frac{9G_m^2}{2} + \frac{9K_m^2}{2} + \frac{9I_m^2}{2} + \frac{9F_m^2}{2} + \frac{5}{6})\}, \end{aligned} \tag{36}$$

and then from (29) we may derive

$$J' - \mathcal{K}J \leq D(t), \tag{37}$$

where we have introduced the function $J(t)$ defined by

$$J(t) = \int_0^t (\frac{1}{4} \| T \|^2 + \frac{5}{16} \| C \|^2 + \frac{1}{4} \| T \|^4 + \frac{1}{6} \| T \|^6) ds. \tag{38}$$

Upon setting

$$D_1 = \int_0^t D(s) \exp(\mathcal{K}(t-s)) ds, \tag{39}$$

one integrates (37) to show

$$J(t) \leq D_1. \tag{40}$$

Upon further setting $D_2(t) = \mathcal{K}D_1 + D$, one uses (39) to find

$$\frac{1}{4} \| T \|^2 + \frac{5}{16} \| C \|^2 + \frac{1}{4} \| T \|^4 + \frac{1}{6} \| T \|^6 \leq D_2(t). \tag{41}$$

Thus, (41) and (40) yield

$$\begin{aligned} \| T \|^2 \leq 4D_2, \quad \| C \|^2 \leq \frac{16}{5} D_2, \quad \| T \|^4 \leq 4D_2, \quad \| T \|^6 \leq 6D_2, \\ \int_0^t \| T \|^2 \leq 4D_1, \quad \int_0^t \| C \|^2 \leq \frac{16}{5} D_1, \quad \int_0^t \| T \|^4 \leq 4D_1, \quad \int_0^t \| T \|^6 \leq 6D_1. \end{aligned} \tag{42}$$

Furthermore, from inequality (29) we then find

$$\begin{aligned} \int_0^t \| \nabla T \|^2 \leq 4D_2, \quad \int_0^t \| \nabla C \|^2 \leq \frac{4}{3} D_2, \\ \int_0^t \| \nabla T^2 \|^2 \leq \frac{4}{3} D_2, \quad \int_0^t \| \nabla T^3 \|^2 \leq \frac{9}{5} D_2. \end{aligned} \tag{43}$$

The next step is to derive a bound for $\sup_{\Omega \times [0, T]} |T|$. To this end, we form the combination

$$\int_0^t \int_{\Omega} (T^{2p-1} - H)(T_s + v_i T_i - \Delta T - Q) dx ds = 0. \tag{44}$$

After some integrations by parts we then show

$$\begin{aligned} \frac{1}{2p} \int_0^t \int_{\Omega} \frac{d}{ds} (T^{2p}) dx ds - \int_0^t \int_{\Omega} T^{2p-1} \Delta T dx ds - \int_0^t \int_{\Omega} T^{2p-1} Q dx ds - (H, T) + (H_0, T_0) \\ - \int_0^t \int_{\Omega} H v_i T_{,i} dx ds + \int_0^t (H_{,s}, T) ds + \int_0^t \int_{\Omega} H \Delta T dx ds + \int_0^t \int_{\Omega} H Q dx ds = 0 \end{aligned} \tag{45}$$

then

$$\begin{aligned} \int_{\Omega} T^{2p} dx + \frac{2(2p-1)}{p} \int_0^t \int_{\Gamma} \nabla(T^p) \nabla(T^p) dx ds &= \int_{\Omega} T_0^{2p} dx + 2p(H, T) \\ &- 2p(H_0, T_0) - 2p \int_0^t (H_{,s}, T) ds + 2p \int_0^t \int_{\Gamma} H v_i T_{,i} dx ds + 2p \int_0^t \oint_{\Gamma} h \frac{\partial H}{\partial n} dA ds \\ &+ 2p \int_0^t \int_{\Omega} T^{2p-1} Q dx ds - 2p \int_0^t \int_{\Omega} H Q dx ds. \end{aligned} \quad (46)$$

Now, by using the maximum principle, arithmetic-geometric mean inequality and bound (15)₁ we have

$$\begin{aligned} 2p \int_0^t \int_{\Omega} H v_i T_{,i} dx ds &\leq 2p H_m \sqrt{\int_0^t \| \mathbf{v} \|^2 ds \int_0^t \| \nabla T \|^2 ds} \\ &\leq 2p \frac{3H_m}{\sqrt{2}} \left(\sqrt{\int_0^t (\| T \|^2 + \| C \|^2 + \| T \|^4 + \| T \|^6) ds} \right) \sqrt{\int_0^t \| \nabla T \|^2 ds}, \end{aligned} \quad (47)$$

where H_m denotes the maximum value of H on Γ . Then with the aid of the Cauchy-Schwarz and the arithmetic-geometric mean inequalities one sees that

$$-2p \int_0^t (H_{,s}, T) ds \leq 2p \sqrt{\int_0^t \| H_{,s} \|^2 ds \int_0^t \| T \|^2 ds}, \quad (48)$$

$$-2p \int_0^t \oint_{\Gamma} h \frac{\partial H}{\partial n} dA ds \leq 2p \sqrt{\int_0^t \oint_{\Gamma} h^2 dA ds \int_0^t \oint_{\Gamma} \left(\frac{\partial H}{\partial n}\right)^2 dA ds}, \quad (49)$$

$$2p \int_0^t \int_{\Omega} T^{2p-1} Q dx ds \leq \int_0^t \int_{\Omega} Q^{2p} ds + (2p-1) \int_0^t \int_{\Omega} T^{2p} dx ds, \quad (50)$$

and

$$2p \int_0^t \int_{\Omega} H Q dx ds \leq \int_0^t \int_{\Omega} Q^{2p} ds + (2p-1) \int_0^t \int_{\Omega} H^{2p/(2p-1)} dx ds. \quad (51)$$

Then use of (47)-(51) in (46) leads to

$$\begin{aligned} \int_{\Omega} T^{2p} dx &\leq \int_{\Omega} T_0^{2p} dx + 2p(\| T \| \| H \| + \| T_0 \| \| H_0 \|) + 2p \sqrt{\int_0^t \| H_{,s} \|^2 ds \int_0^t \| T \|^2 ds} \\ &+ \frac{3}{\sqrt{2}} p h_m^{2p-1} \sqrt{\int_0^t \| T \|^2 ds + \int_0^t \| C \|^2 ds + \int_0^t \| T \|^4 ds + \int_0^t \| T \|^6 ds} \sqrt{\int_0^t \| \nabla T \|^2 ds} \\ &+ 2p \sqrt{\int_0^t \oint_{\Gamma} h^2 dA ds \int_0^t \oint_{\Gamma} \left(\frac{\partial H}{\partial n}\right)^2 dA ds} + 2 \int_0^t \int_{\Omega} Q^{2p} dx ds \\ &+ (2p-1) \int_0^t \int_{\Omega} T^{2p} dx ds + (2p-1) \int_0^t \int_{\Omega} H^{2p/(2p-1)} dx ds. \end{aligned} \quad (52)$$

An application of inequalities (33) and (32) together with (42) and (43) yields

$$\begin{aligned} \int_{\Omega} T^{2p} dx &\leq \int_{\Omega} T_0^{2p} dx + 2p(\sqrt{4D_2} + \| T_0 \|) \psi_1^{1/2} (\oint_{\Gamma} h^{4p-2} dA)^{1/2} \\ &+ 2p \sqrt{4D_1 \psi_1 \int_0^t \oint_{\Gamma} [(h^{2p-1})_{,s}]^2 dA ds} + \frac{3}{2} p h_m^{2p-1} \sqrt{(22D_1 + 4D_2)} \\ &+ 2p \sqrt{\int_0^t \oint_{\Gamma} h^2 dA ds \left(\frac{c_2}{c_1}\right) \int_0^t \oint_{\Gamma} |\nabla_s h^{2p-1}|^2 dA ds} + 2 \int_0^t \| Q \|_{2p}^{2p} ds \\ &+ (2p-1) \int_0^t \int_{\Omega} T^{2p} dx ds + \psi_1 (2p-1) \int_0^t \int_{\Omega} h^{2p} dx ds. \end{aligned} \quad (53)$$

Then, a further application of the Cauchy-Schwarz leads to

$$\left(\oint_{\Gamma} h^{4p-2} dA\right)^{1/2} \leq h_m^{2p-1} \left(\oint_{\Gamma} dA\right)^{1/2} = \frac{h_m^{2p}}{h_m} [m(\Gamma)]^{1/2}, \quad (54)$$

$$\left(\int_0^t \oint_{\Gamma} h^{4p-4} h_s^2 dAds\right)^{1/2} \leq h_m^{2p-2} \left(\int_0^t \oint_{\Gamma} h_s^2 dAds\right)^{1/2} = \frac{h_m^{2p}}{h_m^2} \left(\int_0^t \oint_{\Gamma} h_s^2 dAds\right)^{1/2}, \quad (55)$$

and

$$\left(\int_0^t \oint_{\Gamma} h^{4p-4} |\nabla_s h|^2 dAds\right)^{1/2} \leq h_m^{2p-2} \left(\int_0^t \oint_{\Gamma} |\nabla_s h|^2 dAds\right)^{1/2} = \frac{h_m^{2p}}{h_m^2} \left(\int_0^t \oint_{\Gamma} |\nabla_s h|^2 dAds\right)^{1/2}, \quad (56)$$

where $m(\Gamma)$ is the surface measure of Γ . By employing (54)-(56) in (53), we thus derive

$$\begin{aligned} \int_{\Omega} T^{2p} d\mathbf{x} &\leq \int_{\Omega} T_0^{2p} d\mathbf{x} + 2 \int_0^t \|Q\|^{2p} ds + (2p-1) \int_0^t \int_{\Omega} T^{2p} d\mathbf{x} ds \\ &+ \frac{2p}{h_m} (\sqrt{4D_1} + \|T_0\|) \psi_1^{1/2} h_m^{2p} [m(\Gamma)]^{1/2} + \frac{2p}{h_m} (4D_1 \psi_1)^{1/2} (2p-1) h_m^{2p} \left(\int_0^t \oint_{\Gamma} h_s^2 dAds\right)^{1/2} \\ &+ \frac{3p(22D_1+4D_2)^{1/2}}{2h_m} h_m^{2p} + \psi_1 (2p-1) t m(\Gamma) h_m^{2p} \\ &+ \frac{2p(2p-1)}{h_m^2} \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \left(\int_0^t \oint_{\Gamma} |\nabla_s h|^2 dAds\right)^{\frac{1}{2}} \left(\int_0^t \oint_{\Gamma} h^2 dAds\right)^{\frac{1}{2}} h_m^{2p}. \end{aligned} \quad (57)$$

Denote the coefficients of the first to sixth terms in parentheses, on the right of (58), by $r_1(p), \dots, r_6(p)$, then

$$\begin{aligned} \int_{\Omega} T^{2p} d\mathbf{x} &\leq \int_{\Omega} T_0^{2p} d\mathbf{x} + (r_2 + r_3 + r_4 + r_5 + r_6) h_m^{2p} \\ &+ 2 \int_0^t \|Q\|_{2p}^{2p} ds + \psi_1 t m(\Gamma) h_m^{2p} (2p-1) + (2p-1) \int_0^t \oint_{\Gamma} T^{2p} d\mathbf{x} ds. \end{aligned} \quad (58)$$

Let $\mathcal{B}(p)$ has the form

$$\mathcal{B}(p) = \int_{\Omega} T_0^{2p} d\mathbf{x} + (r_2 + r_3 + r_4 + r_5 + r_6) h_m^{2p} + 2 \int_0^t \|Q\|_{2p}^{2p} ds + \psi_1 t m(\Gamma) h_m^{2p} (2p-1), \quad (59)$$

then we find

$$\int_{\Omega} T^{2p} d\mathbf{x} \leq (2p-1) \int_0^t \oint_{\Gamma} T^{2p} d\mathbf{x} ds + \mathcal{B}(p). \quad (60)$$

Inequality (60) is now integrated, and then we take the $1/2p$ power, to find

$$[R'(t)]^{1/2p} \leq \left[\int_0^t \mathcal{B}(p) e^{(2p-1)(t-s)} ds\right]^{1/2p}. \quad (61)$$

Let now $p \rightarrow \infty$ and then (61) leads to

$$\sup_{\Omega \times [0, \mathcal{T}]} |T| \leq \max\{|T_0|_m, \sup_{[0, \mathcal{T}]} |Q|_m, \sup_{[0, \mathcal{T}]} |h_m|\}. \quad (62)$$

Lemma 3.1 *If $C(x, 0) \in L^\infty(\Omega)$, then*

$$\sup_{\Omega \times [0, \mathcal{T}]} |C| \leq \max\{|C_0|_m\}. \quad (63)$$

Proof: Multiply (4) by c^{p-1} for $p > 1$ (where we assume the concentration is scaled to be non-negative, otherwise p is chosen as an even integer). Thus,

$$\frac{d}{dt} \int_{\Omega} C^p d\mathbf{x} = -p(p-1) \int_{\Omega} C^{p-2} |\nabla C|^2 d\mathbf{x} - K_1 p \int_{\Omega} C^p d\mathbf{x}. \quad (64)$$

We may integrate this and drop non-positive terms on the right to deduce

$$\left[\int_{\Omega} C^p d\mathbf{x}\right]^{1/p} \leq \left[\int_{\Omega} C_0^p d\mathbf{x}\right]^{1/p}. \quad (65)$$

Let now $p \rightarrow \infty$ in (65) to find the desired result. \blacksquare

Inequality (62) and (63) are a priori estimates we are seeking for T and C , respectively. We henceforth denote the right-hand side of (62) and (63) by T_m and C_m , respectively.

4. Continuous dependence on Q

Let $\{u_i, S, \varphi, \pi_1\}$ and $\{v_i, T, C, \pi_2\}$ be solutions to \mathcal{P} for the same values of h, k, T_0 and C_0 but for different heat source functions $R(\mathbf{x}, t)$ and $Q(\mathbf{x}, t)$. Define the difference variables

w_i, θ, ϕ, q and Π by

$$w_i = u_i - v_i, \theta = S - T, \phi = \varphi - C, q = R - Q, \Pi = \pi_1 - \pi_2.$$

It is easily verified that $\{w_i, \theta, \phi, \pi\}$ satisfies the boundary-initial value problem

$$\begin{aligned} w_i + a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i] &= -\Pi_{,i} + g_i\theta + h_i(S + T)\theta \\ &+ L_i\theta(S^2 + ST + T^2) + I_i\phi, \\ w_{i,i} &= 0, \\ \theta_{,t} + w_iT_{,i} + u_i\theta_{,i} &= \Delta\theta + q, \\ \phi_{,t} + w_iC_{,i} + u_i\phi_{,i} &= \Delta\phi. \end{aligned} \tag{66}$$

on $\Omega \times (0, T]$, together with

$$\begin{aligned} w_i n_i = 0, \theta = 0, \phi = 0, & \quad \text{on } \Gamma \times (0, T], \\ \theta(\mathbf{x}, 0) = 0, \phi(\mathbf{x}, 0) = 0, & \quad \mathbf{x} \in \Omega, \end{aligned} \tag{67}$$

Multiply (66)₁ by w_i and integrate over Ω . Using the Cauchy-Schwarz inequality and arithmetic-geometric mean inequality one obtains

$$\begin{aligned} &\|\mathbf{w}\|^2 + a \int_{\Omega} [|\mathbf{u}|u_i - |\mathbf{v}|v_i]w_i d\mathbf{x} + b \int_{\Omega} [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i d\mathbf{x} \\ &= (\theta g_i, w_i) + (h_i(S + T)\theta, w_i) + (L_i\theta(S^2 + TS + T^2)\phi, w_i) + (I_i\phi, w_i) \\ &\leq (\theta, w_i) + (S_m + T_m)(\theta, w_i) + (S_m^2 + T_m S_m + T_m^2)(\theta, w_i) + (\phi, w_i) \\ &= (\theta, w_i) + X_m(\theta, w_i) + Y_m(\theta, w_i) + (\phi, w_i) \\ &\leq \frac{1}{6} \|\mathbf{w}\|^2 + \frac{3}{2} \|\theta\|^2 + \frac{1}{6} \|\mathbf{w}\|^2 + \frac{3}{2} X_m^2 \|\theta\|^2 + \frac{1}{6} \|\mathbf{w}\|^2 + \frac{3}{2} Y_m^2 \|\theta\|^2 \\ &\quad + \frac{1}{6} \|\mathbf{w}\|^2 + \frac{3}{2} \|\phi\|^2 \\ &\leq \frac{2}{3} \|\mathbf{w}\|^2 + \frac{3}{2} (1 + X_m^2 + Y_m^2) [\|\theta\|^2 + \|\phi\|^2], \end{aligned} \tag{68}$$

where $X_m = S_m + T_m$, and $Y_m = S_m^2 + T_m S_m + T_m^2$ with S_m being the maximum value of S , analogous to T_m . Now, we observe with the aid of the triangle inequality, that

$$\begin{aligned} a \int_{\Omega} [|\mathbf{u}|u_i - |\mathbf{v}|v_i]w_i d\mathbf{x} &= \frac{a}{2} \int_{\Omega} [|\mathbf{u}| + |\mathbf{v}|]w_i w_i d\mathbf{x} + \frac{a}{2} \int_{\Omega} [|\mathbf{u}| - |\mathbf{v}|]^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ &\geq \frac{a}{2} \int_{\Omega} [|\mathbf{u}| + |\mathbf{v}|]w_i w_i d\mathbf{x} \geq \frac{a}{2} \int_{\Omega} |\mathbf{u} - \mathbf{v}|w_i w_i d\mathbf{x} \\ &\geq \frac{a}{2} \int_{\Omega} |w_i|w_i w_i d\mathbf{x} = \frac{a}{2} \|\mathbf{w}\|_3^3, \end{aligned} \tag{69}$$

and, similarly

$$\begin{aligned} b \int_{\Omega} [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i d\mathbf{x} &= \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2]w_i w_i d\mathbf{x} + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 - |\mathbf{v}|^2]^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ &\geq \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2]w_i w_i d\mathbf{x} \geq \frac{b}{4} \int_{\Omega} [|\mathbf{u}| + |\mathbf{v}|]^2 w_i w_i d\mathbf{x} \\ &\geq \frac{b}{4} \int_{\Omega} |u_i - v_i|^2 w_i w_i d\mathbf{x} = \frac{b}{4} \|\mathbf{w}\|_4^4. \end{aligned} \tag{70}$$

Next, we put (69) and (70) in (68), to find

$$\|\mathbf{w}\|^2 + \frac{3a}{2} \|\mathbf{w}\|_3^3 + \frac{3b}{4} \|\mathbf{w}\|_4^4 \leq \frac{9}{2} (1 + X_m^2 + Y_m^2) [\|\theta\|^2 + \|\phi\|^2]. \tag{71}$$

Furthermore, multiply (3) and (4) by θ and ϕ , respectively, and integrate over Ω to see after integrated by parts, discard the negative terms, and used a little rearrangement that

$$\frac{d}{dt} \|\theta\|^2 \leq \|q\|^2 + \frac{T_m^2}{2} \|\mathbf{w}\|^2, \tag{72}$$

$$\frac{d}{dt} \|\phi\|^2 \leq \frac{C_m^2}{2} \|\mathbf{w}\|^2, \tag{73}$$

where C_m being the maximum value of C . From (72) and (73), we have

$$\frac{d}{dt} [\|\theta\|^2 + \|\phi\|^2] \leq \|q\|^2 + \frac{1}{2} (C_m^2 + T_m^2) \|\mathbf{w}\|^2. \tag{74}$$

Upon insertion of (71) in (74) we see that

$$\frac{d}{dt} [\|\theta\|^2 + \|\phi\|^2] \leq \|q\|^2 + \frac{9}{4}(C_m^2 + T_m^2)(1 + X_m^2 + Y_m^2)[\|\theta\|^2 + \|\phi\|^2]. \quad (75)$$

Let

$$\mathbf{K} = \frac{9}{4}(C_m^2 + T_m^2)(1 + X_m^2 + Y_m^2),$$

then, upon integration of (75) we find

$$\|\theta\|^2 + \|\phi\|^2 \leq \|q\|^2 \left[\frac{\exp(\mathbf{K}t) - 1}{\mathbf{K}} \right]. \quad (76)$$

Thus, we have established continuous dependence on Q in the L^2 measure of θ and ϕ . Additionally, from inequalities (76) in (71) we then derive

$$\|\mathbf{w}\|^2 + \frac{3a}{2} \|\mathbf{w}\|_3^3 + \frac{3b}{4} \|\mathbf{w}\|_4^4 \leq \frac{9}{2}(1 + X_m^2 + Y_m^2) \|q\|^2 \left[\frac{\exp(\mathbf{K}t) - 1}{\mathbf{K}} \right]. \quad (77)$$

Therefore, we also have continuous dependence on Q in the L^2, L^3 and L^4 measures of \mathbf{w} .

5. Continuous dependence on a

Now, to investigate continuous dependence on a , we let (u_i, S, ϕ, π_1) and (v_i, T, C, π_2) be solutions to (1)-(4) for the same boundary-initial-value problems for different coefficients a_1 and a_2 . Define the difference variables and constant as

$$w_i = u_i - v_i, \theta = S - T, \phi = \phi - C, \Pi = \pi_1 - \pi_2, a = a_1 - a_2. \quad (78)$$

Then (w_i, θ, ϕ, Π) solves the boundary-initial-value problem

$$w_i + [a_1|\mathbf{u}|u_i - a_2|\mathbf{v}|v_i] + b[|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i] = -\Pi_{,i} + g_i\theta + h_i(S + T)\theta + L_i\theta(S^2 + ST + T^2) + I_i\phi, \quad (79)$$

$$w_{i,i} = 0, \quad (80)$$

$$\theta_{,t} + w_iT_{,i} + u_i\theta_{,i} = \Delta\theta, \quad (81)$$

$$\phi_{,t} + w_iC_{,i} + u_i\phi_{,i} = \Delta\phi. \quad (82)$$

To establish continuous dependence on a we rearrange the a_1 and a_2 terms as

$$a_1|\mathbf{u}|u_i - a_2|\mathbf{v}|v_i = \frac{a}{2}[|\mathbf{u}|u_i + |\mathbf{v}|v_i] + \tilde{a}[|\mathbf{u}|u_i - |\mathbf{v}|v_i], \quad (83)$$

where $\tilde{a} = \frac{a_1 + a_2}{2}$. Hence, from (83) and (79) we arrive at

$$w_i + \frac{a}{2}[|\mathbf{u}|u_i + |\mathbf{v}|v_i] + \tilde{a}[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i] = -\Pi_{,i} + g_i\theta + h_i(S + T)\theta + L_i\theta(S^2 + ST + T^2) + I_i\phi. \quad (84)$$

Furthermore, from (84), the Cauchy-Schwarz inequality and arithmetic-geometric mean inequality we may derive

$$\begin{aligned} & \|\mathbf{w}\|^2 + \frac{a}{2} \int_{\Omega} [|\mathbf{u}|u_i + |\mathbf{v}|v_i]w_i dx + \tilde{a} \int_{\Omega} [|\mathbf{u}|u_i - |\mathbf{v}|v_i]w_i dx \\ & + b \int_{\Omega} [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i dx \\ & = (g_i\theta, w_i) + (h_i(S + T)\theta, w_i) + (L_i\theta(S^2 + ST + T^2), w_i) + (I_i\phi, w_i) \\ & \leq \frac{2}{3} \|\mathbf{w}\|^2 + \frac{3}{2}(1 + X_m^2 + Y_m^2)[\|\theta\|^2 + \|\phi\|^2], \end{aligned} \quad (85)$$

since, from (69) we have

$$\tilde{a} \int_{\Omega} [|\mathbf{u}|u_i - |\mathbf{v}|v_i]w_i dx = \frac{\tilde{a}}{2} \int_{\Omega} [|\mathbf{u}| + |\mathbf{v}|]w_i w_i dx + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}|^2 - |\mathbf{v}|^2)[|\mathbf{u}| + |\mathbf{v}|] dx, \quad (86)$$

and from (70), we obtain

$$b \int_{\Omega} [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i dx \geq \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2]w_i w_i dx + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 - |\mathbf{v}|^2]^2 dx, \quad (87)$$

then, we use (86) and (87) in (85), to find

$$\begin{aligned} & \frac{1}{3} \| \mathbf{w} \|^2 + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}| + |\mathbf{v}|) w_i w_i d\mathbf{x} + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}| - |\mathbf{v}|)^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ & + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x} + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 - |\mathbf{v}|^2]^2 d\mathbf{x} \\ & \leq \frac{3}{2} (1 + X_m^2 + Y_m^2) [\| \theta \|^2 + \| \phi \|^2] - \frac{a}{2} \int_{\Omega} [|\mathbf{u}| u_i w_i + |\mathbf{v}| v_i w_i] d\mathbf{x}. \end{aligned} \tag{88}$$

From this point the proof splits into two parts depending on whether $b > 0$ or $b = 0$. Suppose $b = 0$, we then use the Cauchy-Schwarz and arithmetic-geometric mean inequalities to see that

$$\frac{a}{2} \left| \int_{\Omega} [|\mathbf{u}| u_i w_i + |\mathbf{v}| v_i w_i] d\mathbf{x} \right| \leq \frac{a^2}{8b^2} \int_{\Omega} (u_i u_i + v_i v_i) d\mathbf{x} + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x}. \tag{89}$$

If we use (89) in (88), we find

$$\begin{aligned} \frac{1}{3} \| \mathbf{w} \|^2 & \leq \frac{3}{2} (1 + X_m^2 + Y_m^2) [\| \theta \|^2 + \| \phi \|^2] + \frac{a^2}{8b^2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2] d\mathbf{x} \\ & = \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + \frac{a^2}{8b^2} \| \mathbf{u} \|^2 + \frac{a^2}{8b^2} \| \mathbf{v} \|^2, \end{aligned} \tag{90}$$

where

$$\mathbf{K}_1 = \frac{3}{2} (1 + X_m^2 + Y_m^2).$$

Now, we will find bounds for $\| \mathbf{u} \|^2$ and $\| \mathbf{v} \|^2$. By multiplying (1) by v_i and and integrating over Ω , we then obtain

$$\begin{aligned} \| \mathbf{v} \|^2 + a \| \mathbf{v} \|^3 + b \| \mathbf{v} \|^4 & \leq (T, v_i) + (T^2, v_i) + (T^3, v_i) + (C, v_i) \\ & \leq \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|^2 + \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|^4 + \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|^6 + \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| C \|^2, \end{aligned} \tag{91}$$

which lead to the following bound

$$\frac{1}{3} \| \mathbf{v} \|^2 + a \| \mathbf{v} \|^3 + b \| \mathbf{v} \|^4 \leq \mathbf{K}_2, \tag{92}$$

where $\mathbf{K}_2 = \frac{3}{2} \| T \|^4 + \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| T \|^6 + \frac{1}{6} \| \mathbf{v} \|^2 + \frac{3}{2} \| C \|^2$. Thus, we have the following estimates

$$\| \mathbf{v} \|^2 \leq 3\mathbf{K}_2 = \mathbf{K}_3, \quad \| \mathbf{v} \|^3 \leq \frac{\mathbf{K}_2}{a} = \mathbf{K}_4, \quad \| \mathbf{v} \|^4 \leq \frac{\mathbf{K}_2}{b} = \mathbf{K}_5, \tag{93}$$

and similarly we have

$$\| \mathbf{u} \|^2 \leq 3\mathbf{K}_2 = \mathbf{K}_3, \quad \| \mathbf{u} \|^3 \leq \frac{\mathbf{K}_2}{a} = \mathbf{K}_4, \quad \| \mathbf{u} \|^4 \leq \frac{\mathbf{K}_2}{b} = \mathbf{K}_5. \tag{94}$$

Then, from (90), (93) and (94) we find

$$\frac{1}{3} \| \mathbf{w} \|^2 \leq \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + a^2 \mathbf{K}_6, \tag{95}$$

where $\mathbf{K}_6 = \frac{\mathbf{K}_3}{4b^2}$.

Suppose now that $b = 0$, we then have from (88) that

$$\begin{aligned} & \frac{1}{3} \| \mathbf{w} \|^2 + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}| + |\mathbf{v}|) w_i w_i d\mathbf{x} + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}| - |\mathbf{v}|)^2 (|\mathbf{u}| + |\mathbf{v}|) d\mathbf{x} \\ & \leq \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] - \frac{a}{2} \int_{\Omega} [|\mathbf{u}| u_i w_i + |\mathbf{v}| v_i w_i] d\mathbf{x}. \end{aligned} \tag{96}$$

We then use the Cauchy-Schwarz and arithmetic-geometric mean inequalities as follows:

$$\frac{a}{2} \left| \int_{\Omega} [|\mathbf{u}| u_i w_i + |\mathbf{v}| v_i w_i] d\mathbf{x} \right| \leq \frac{a^2}{8\tilde{a}} \int_{\Omega} (|\mathbf{u}|^3 + |\mathbf{v}|^3) d\mathbf{x} + \frac{\tilde{a}}{2} \int_{\Omega} (|\mathbf{u}| + |\mathbf{v}|) w_i w_i d\mathbf{x}. \tag{97}$$

We now employ (97) in (96) to find, after discard the positive term, that

$$\frac{1}{3} \| \mathbf{w} \|^2 \leq \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + \frac{a^2}{8\tilde{a}} \int_{\Omega} (|\mathbf{u}|^3 + |\mathbf{v}|^3) d\mathbf{x}. \tag{98}$$

By inserting the estimates (93) and (94) in (98), we find

$$\frac{1}{3} \| \mathbf{w} \|^2 \leq \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + a^2 \mathbf{K}_7,$$

where $\mathbf{K}_7 = \frac{\mathbf{K}_4}{4a}$. Thus, we have for both cases $b > 0$ and $b = 0$, that

$$\frac{1}{3} \|\mathbf{w}\|^2 \leq \mathbf{K}_1 [\|\theta\|^2 + \|\phi\|^2] + a^2 \mathbf{K}_8, \tag{99}$$

where $\mathbf{K}_8 = \max\{\mathbf{K}_6, \mathbf{K}_7\}$. Thus, we find

$$\|\mathbf{w}\|^2 \leq \mathbf{K}_1 [\|\theta\|^2 + \|\phi\|^2] + 3a^2 \mathbf{K}_8. \tag{100}$$

Also, it is easy to show that

$$\begin{aligned} \frac{d}{dt} \|\theta\|^2 &\leq \frac{T_m^2}{2} \|\mathbf{w}\|^2, \\ \frac{d}{dt} \|\phi\|^2 &\leq \frac{C_m^2}{2} \|\mathbf{w}\|^2, \end{aligned} \tag{101}$$

which lead to

$$\frac{d}{dt} [\|\theta\|^2 + \|\phi\|^2] \leq \frac{1}{2} (T_m^2 + C_m^2) \|\mathbf{w}\|^2. \tag{102}$$

Use of (100) in (102) and then integration the resulted equation allows us to see that

$$\|\theta\|^2 + \|\phi\|^2 \leq a^2 \mathbf{K}_{10} \left[\frac{\exp(\mathbf{K}_9 t) - 1}{\mathbf{K}_9} \right], \tag{103}$$

where

$$\mathbf{K}_9 = \frac{3}{2} (T_m^2 + C_m^2) \mathbf{K}_1,$$

and

$$\mathbf{K}_{10} = \frac{3}{2} (T_m^2 + C_m^2) \mathbf{K}_8,$$

Inequality (103) establishes continuous dependence of θ and ϕ in L^2 on the Forchheimer coefficient a . A similar continuous dependence estimate for w_i may then be established with the help of (100) and (103).

6. Continuous dependence on b

We commence with a study of continuous dependence on the coefficient b . Therefore, let (u_i, S, φ, π_1) and (v_i, T, C, π_2) be solutions to Equations (1)-(4) for the same boundary and initial conditions, but for different coefficient b_1 and b_2 . Define the difference variables u_i, θ, ϕ and Π and constant b by

$$w_i = u_i - v_i, \quad \theta = S - T, \quad \phi = E - c, \quad \Pi = \pi_1 - \pi_2, \quad b = b_1 - b_2, \tag{104}$$

and then we find that (w_i, θ, ϕ, π) satisfy the boundary-initial value problem

$$\begin{aligned} w_i + a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + [b_1|\mathbf{u}|^2u_i - b_2|\mathbf{v}|^2v_i] &= -\Pi_i + g_i\theta + h_i(S + T)\theta \\ &+ L_i\theta(S^2 + ST + T^2) + I_i\phi, \\ w_{i,i} &= 0, \\ \theta_{,t} + w_iT_{,i} + u_i\theta_{,i} &= \Delta\theta, \\ \phi_{,t} + w_iC_{,i} + u_i\phi_{,i} &= \Delta\theta. \end{aligned} \tag{105}$$

We write

$$b_1|\mathbf{u}|^2u_i - b_2|\mathbf{v}|^2v_i = \frac{b}{2} [|\mathbf{u}|^2u_i + |\mathbf{v}|^2v_i] + \tilde{b}[|\mathbf{u}|u_i - |\mathbf{v}|v_i], \tag{106}$$

where $\tilde{b} = (b_1 + b_2)/2$. Multiply (105)₁ by w_i and integrate over Ω , and then use (106) and (69) in the resulting equation from (105)₁ to find

$$\begin{aligned} \|\mathbf{w}\|^2 + \frac{a}{2} \int_{\Omega} (|\mathbf{u}| + |\mathbf{v}|)w_iw_i dx + \frac{a}{2} \int_{\Omega} (|\mathbf{u}|^2 - |\mathbf{v}|^2)(|\mathbf{u}| + |\mathbf{v}|) dx \\ + \frac{b}{2} \int_{\Omega} [|\mathbf{u}|^2u_iw_i + |\mathbf{v}|^2v_iw_i] dx + \tilde{b} \int_{\Omega} [|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i]w_i dx \\ \leq \frac{2}{3} \|\mathbf{w}\|^2 + \frac{3}{2} (1 + X_m^2 + Y_m^2) [\|\theta\|^2 + \|\phi\|^2]. \end{aligned} \tag{107}$$

We use (87) for \tilde{b} in (107) then discard the a terms from the resulting equation to find

$$\frac{1}{3} \| \mathbf{w} \|^2 \leq \frac{3}{2} (1 + X_m^2 + Y_m^2) [\| \theta \|^2 + \| \phi \|^2] - \frac{b}{2} \int_{\Omega} [|\mathbf{u}|u_i w_i + |\mathbf{v}|v_i w_i] d\mathbf{x} - \frac{\tilde{b}}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x}. \tag{108}$$

Next, the Cauchy-Schwarz and arithmetic-geometric mean inequalities are employed to see that

$$\frac{b}{2} \left| \int_{\Omega} [|\mathbf{u}|^2 u_i w_i + |\mathbf{v}|^2 v_i w_i] d\mathbf{x} \right| \leq \frac{b^2}{8\tilde{b}} \int_{\Omega} (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x} + \frac{\tilde{b}}{2} \int_{\Omega} [|\mathbf{u}|^2 + |\mathbf{v}|^2] w_i w_i d\mathbf{x}. \tag{109}$$

Substituting (109) in (108) to obtain

$$\begin{aligned} \frac{1}{3} \| \mathbf{w} \|^2 &\leq \frac{3}{2} (1 + X_m^2 + Y_m^2) [\| \theta \|^2 + \| \phi \|^2] + \frac{b^2}{8\tilde{b}} \int_{\Omega} (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x} \\ &\leq \mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + \frac{b^2}{8\tilde{b}} \int_{\Omega} (|\mathbf{u}|^4 + |\mathbf{v}|^4) d\mathbf{x}. \end{aligned} \tag{110}$$

We next employ the estimates (93) and (94) in (110) to arrive at

$$\frac{1}{3} \| \mathbf{w} \|^2 \leq 3\mathbf{K}_1 [\| \theta \|^2 + \| \phi \|^2] + 3b^2 \mathbf{K}_{10}, \tag{111}$$

where $\mathbf{K}_{10} = \frac{1}{4\tilde{b}} \mathbf{K}_5$. Next, by using the above bound in (102), and after integration we find

$$\| \theta \|^2 + \| \phi \|^2 \leq b^2 \mathbf{K}_{12} \left[\frac{\exp(\mathbf{K}_{11}t) - 1}{\mathbf{K}_{11}} \right], \tag{112}$$

where $\mathbf{K}_{11} = \frac{3}{2} (T_m^2 + C_m^2) \mathbf{K}_1$ and $\mathbf{K}_{12} = \frac{3}{2} (T_m^2 + C_m^2) \mathbf{K}_{10}$. The continuous dependence for w_i on the Forchheimer coefficient b follows directly from (111) and (112).

7. Conclusions

The equations for double diffusive convection in a porous medium of Forchheimer's law are analysed when the density of fluid depends on temperature and concentration as a cubic and linear function. The question of continuous dependence of the solution on the heat source and Forchheimer coefficients is one which belongs to the general class of structural stability problems. Structural stability (or continuous dependence on the model itself) is one of major importance and it may be argued that it is as or more important than the widely accepted notion of stability as continuous dependence upon the initial data. Therefore, we establish rigorous a priori bounds with coefficients which depend only on boundary data and initial data and which demonstrate continuous dependence of the solution on changes in Q, a and b .

References

- [1] B. Straughan, Stability and Wave Motion in Porous Media, in: Series in Appl. Math Sci., vol. 165, Springer, 2008.
- [2] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, New York, 1974.
- [3] R.J. Knops, L.E. Payne, Continuous data dependence for the equations of classical elastodynamics, Proc. Camb. Phil. Soc. 66 (1969) 481-491.
- [4] R.J. Knops, L.E. Payne, Improved estimates for continuous data dependence in linear elastodynamics, Proc. Camb. Phil. Soc. 103 (1988) 535-559.
- [5] L.E. Payne, On stabilizing ill-posed problems against errors in geometry and modeling, in: H. Engel, C.W. Groetsch (Eds.), Proc. of the Conference on Inverse and Ill-Posed Problems: Stroblh, Academic Press, New York, 1987, pp. 399-416.
- [6] L.E. Payne, On geometric and modeling perturbations in partial differential equation, in: R.J. Knops, A.A. Lacey (Eds.), Proc. of the LMS Symposium on Non-Classical Continuum

Mechanics, Cambridge University Press, Cambridge, 1987, pp. 108-128.

- [7] L.E. Payne, Continuous dependence on geometry with applications in continuum mechanics, in: G.A.C. Graham, S.K. Malik (Eds.), *Continuum Mechanics and its Applications*, Hemisphere Publ. Co., Washington DC, 1989, pp. 877-890.
- [8] E. Aulisa, L. Blosanskaya, L. Hoang, A. Ibragimov, Analysis of generalized Forchheimer flows of compressible fluids in porous media, *J. Math. Phys.* 50 (2009) 103102.
- [9] M. Ciarletta, B. Straughan, V. Tibullo, Modelling boundary and nonlinear effects in porous media flow, *Nonlinear Anal. Real World Appl.* 12 (2011) 2839-2843.
- [10] L. Hoang, A. Ibragimov, Structural stability of generalized Forchheimer equations for compressible fluids in porous media, *Nonlinearity* 24 (2011) 1-41.
- [11] A. J. Harfash, Continuous dependence on the coefficients for double diffusive convection in Darcy flow with Magnetic field effect, *Analysis and Mathematical Physics* (2013) 3 163-181.
- [12] A. J. Harfash, Structural stability for convection models in a reacting porous medium with magnetic field effect, *Ricerche di Matematica* 63 (2014) 1-13.
- [13] A. J. Harfash, Structural stability for two convection models in a reacting fluid with magnetic field effect, *Annales Henri Poincare* 15 (2014) 2441-2465.
- [14] Y. Liu, Convergence and continuous dependence for the Brinkman–Forchheimer equations, *Math. Comput. Modelling* 49 (2009) 1401-1415.
- [15] Y. Liu, Y. Du, C. Lin, Convergence results for Forchheimer’s equations for fluid flow in porous media, *J. Math. Fluid Mech.* 12 (2010) 576-593.
- [16] Y. Liu, Y. Du, C. Lin, Convergence and continuous dependence results for the Brinkman equations, *Appl. Math. Comput.* 215 (2010) 4443-4455.
- [17] B. Straughan, Continuous dependence on the heat source in resonant porous penetrative convection, *Stud. Appl. Math.* 127 (2011) 302-314.
- [18] M. Gentile, B. Straughan, Structural stability in resonant penetrative convection in a Forchheimer porous material. *Nonlinear Anal. Real World Appl.* 14 (2013) 397–401.
- [19] G. McKay, B. Straughan, A nonlinear analysis of convection near the density maximum, *Acta Mech.* 95 (1992) 9-28.
- [20] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, second ed., in: *Applied Mathematical Sciences*, vol. 91, Springer-Verlag, New York, 2004.
- [21] P. Forchheimer, *Wasserbewegung durch Boden*, *Z. Vereines Deutscher Ingenieure* 50, 1781-1788 (1901).
- [22] D. A. Nield, A. Bejan, *Convection in Porous Media. 4th ed.*, Springer-Verlag, New York, (2013).
- [23] M.C. Néel, Convection forcée en milieux poreux: écart à la loi de Darcy, *C.R. Acad. Sci. Paris, Série Iib* 326 (1998) 615-620.
- [24] B. Straughan, Resonant porous penetrative convection, *Proc. R. Soc. Lond. Ser. A* 460 (2004) 2913-2927.
- [25] L. E. Payne, B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modelling questions, *J. Math. Pures et Appl.* 77 (1998) 317–354.
- [26] L. E. Payne, H. F. Weinberger, New bounds for solutions of second order elliptic partial differential equations, *Pacific J. Math.* 8 (1958) 551-573.

الاستمرارية المعتمدة للحل لمسألة الحمل المزدوج الانتشار في وسط مسامي عندما تكون الكثافة معتمدة على درجة الحرارة

عقيل جاسم حرفش, ايات عبد الكريم حميد

قسم الرياضيات ، كلية العلوم ، جامعة البصرة ، البصرة ، العراق

المستخلص

تمت دراسة الاستقرار الهيكلية لمسألة الحمل المزدوج الانتشار في وسط مسامي من نوع فورشايمر ، عندما تعتمد كثافة السائل والتركيز على درجة الحرارة كدوال تكعيبية وخطية ، على التوالي. وقد تبين أن لهذه المشكلة ، مع الحمل الحراري فقط وفي طبقة لانهائية، فان الرنين بين الطبقات الداخلية للمائع يمكن أن يحدث. المشكلة الرئيسية هي مصدر الحرارة الداخلي وقد يؤدي وجوده إلى حصول الحمل المتذبذب في حالة عدم الاستقرار الخطي مسببا حالة الرنين. لذا، في هذه الدراسة ، تم تحليل مشكلة الاستقرار الهيكلي للمسألة من حيث دراسة الاعتماد المستمر للحل على مصدر الحرارة نفسه لنموذج الحمل المزدوج الانتشار في وسط مسامي من نوع فورشايمر. علاوة على ذلك ، تم إظهار الاعتماد المستمر للحل على التغييرات في معاملات فورشايمر.