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Random Fractional Laplace Transform for Solving Random Time-Fractional Heat Equation in an Infinite Medium

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Abstract

The random Laplace and Fourier transforms are very important tools to solve random heat problems. Unfortunately, it is difficult to use these random integral transforms for solving the fractional random heat problems, where the mean square conformable fractional derivative is used to express for the time fractional derivative. Therefore, this work adopts the extension of the random Laplace transform into random fractional Laplace transform in order to solve this kind of heat problems. The stochastic process solution of the fractional random heat in an infinite medium is investigated by using random fractional Laplace transform together with random Fourier transform. The mean and the standard diffusion of the stochastic process solution is computed for different value of fractional order. When $\alpha = 1$, the results show agree with the available results in the references context.

1. Introduction

Over the past two centuries, the heat diffusion equation has played an important role to be a powerful tool for studying and analyzing the conductive transfer and storage of heat in a solid material. In fact, the heat equation is describes the process of the thermal conduction inside the solid material, of fixed size and shape, and study the heat switching with the external space through its boundary. The historical contexts refer that Jean B. J. Fourier (1768-1830) was the first person who find a mathematical formula for the heat equation, at the beginning of the nineteenth century. The Fourier's contribution has made a great influence in the development of many sciences, such as; biological sciences, physical sciences, earth sciences, chemical sciences, and social sciences [2, 24]. Conversely, Albert Einstein concerned on studying the Brownian motion and he found that the Brownian motion is a solution to the Fourier's heat equations in a certain sense. In fact, the Albert Einstein's study of Brownian motion led to an interpretation of Fourier's heat equation to a new type of differential equations, which we define it today as stochastic differential equations [25, 18].

In the last few decade, many authors concern with the applications of differential equations with fractional order in many real life problems. Such types of differential equations is called fractional differential equations (FDEs) [16, 21, 22]. On the other hand, many of researchers mixed between the heat(diffusion) equation and the fractional calculus such that: in (2004) Povstenko et al. who depend on heat conduction equation with time fractional derivative by Caputo derivative to write about theory of thermo elasticity, and in (2013) they used time fractional heat (conduction) equation in a composite medium of two semi- infinite medium when the solution was obtained the Laplace that respect to time [26],[27].To describe the phenomenon of heat (conduction), in(2013) Xu et al. used the fractional Cattaneo heat equation in fractional form semi-infinite medium [32]. In the last years, studying the effect of randomization in the behavior of solving fractional differential equations takes a great attention by many researchers [31, 2, 20, 29, 10]. There are many methods have been used to investigate the problem of random heat diffusion in a finite medium such as; finite difference method [17, 12], finite elements method [19] and random perturbation method [11]. In (2018) M.-C. Casaban et al. study the construction of mean square (m.s.) analytic numerical solution of stochastic processes by random Fourier integral transform [9]. Some researcher mixed between the random equation and the fractional calculus like Burgos at el. in (2015) when they study the random fractional(Caputo) equation with random conditions. Recently, in 2014, M.-C. Casaban et al. solved the random diffusion model using mean square approach to extend the deterministic case in Fourier transform to the random framework[6] . In (2015) M.C. Casaban et al. introduced a new formula of Laplace transform to to solve the stochastic process. This transform is called by the random Laplace trans- forms (RLT). The (RLT) has been used to find an explicit solution for random heat equation in semi-finite medium [7]. On other hand, at the same year Thabet Abdeljawad presented fractional Laplace transform when he can development R. Khalil definition in conformable fractional derivative to solve linear fractional equations [1]. This paper is concerned with the studying of a m.s. solution for the random time-conformable fractional heat diffusion (conduction) in one dimensional infinite medium. In 2015, M. -C. Casaban et al extend the classical Fourier transform method to random framework (random Fourier transform) in order to find an analytic-numerical solution for random diffusion problems in the mean square sense [7]. Also, the random Fourier transform has been used to solve random parabolic partial differential equation[8]. In fact, this extension enables us to find the Fourier transform for a stochastic process. Unfortunately, the random Fourier transform alone is not enough to solve the random time-conformable fractional diffusion problem in an infinite medium

$$T_t^{\alpha} U(t,x) = A\frac{\partial^2 U}{\partial x^2}(t,x), \forall x \in (-\infty,\infty) \text{ and } t>0 \tag{1}$$

$$U(0,x) = \Psi(x,B), \forall x \in (-\infty,\infty). \tag{2}$$

So, we propose the implemented the random fractional Laplace transform together with random Fourier transform to find an analytic stochastic process solution for this problem. Also, an approximate mean and variance of the analytic stochastic process solution is investigated.

2. **Preliminaries**

Elementary and fundamental concepts for this paper will be given and reviewed briefly in this section for completeness purpose:

Definition 2.1 [15, 3]: A σ- algebra F is a family of subsets of the sample space Ω (a set of all possible events in an experiment) such that

1. $\Omega \in F$
2. If $C \in F$, then the complement set $\Omega \backslash C$ in F
3. if $C_1, C_2, \ldots, C_n$ is a sequence of set in F then their union or intersection of count ably many algebra.

Definition 2.2 [15, 3]: A probability space is the triple (Ω, F, P) where Ω is a nonempty set, F is an σ−algebra subsets of Ω and P is probability measure function $P : F \rightarrow [0, 1]$. The probability of C for all $C \in F$, P (C), is required that:

1. $P(\Omega) = 1$
2. (Countable additivity) whenever C_1, C_2, ... is a sequence of disjoint.

Definition 2.3 [14, 28]: Let (Ω, F, P) be a probability space. Then the real random variable

(r.v.) $U : \Omega \rightarrow R$ defined on probability space (Ω, F, P) is called of order $p \geq 1$ (p−r.v.), if $E[(U)^p] < \infty$, where **E**[•] is the expectation operator.

The space $L_p(\Omega)$ of all random variables of order $p \geq 1$ (p− r.v.s) with the norm

$$\| U \|_p = (E[(U)^p])^{\frac{1}{p}} \tag{3}$$

is a Banach space. In particle, when p = 2 the r.v. U is called 2− r.v. if $\mathbf{E}[(U)^2] < \infty$, the space $L_2(\Omega)$ of all (2−r.v.s) with the norm

$$\|U\|_2 = (E[(U)^2])^{\frac{1}{2}} \tag{4}$$

is Banach space and it is Hilbert space with the inner product,

$$(U, V) = \mathbf{E}[UV], \quad U, V \in L_2(\Omega) \tag{5}$$

Similarly the real r.v. U is called fourth order r.v. (or $4-$r.v.) if $E[(U)^4] < +\infty$ The space $L_4(\Omega)$ of all $4-$r.v. endowed with the norm

$$\|U\|_4 = E[(U)^4]^{\frac{1}{4}} \tag{6}$$

is a Banach space [30, 13].

Definition 2.4 [28]: The n-dimensional random vector of a second order r.v.s U_i , i = 1, 2, ..., n defined by

$$U = (U_1, U_2, \ldots, U_n) \tag{7}$$

and constitutes a linear vector space if all equivalent r.v.s are identified. The space $L_2^n(\Omega)$ of all n-dimensional random vectors with the norm $\| U \|_n = \underset{j=3,\ldots,n}{\operatorname{Max}} \| U_j \|_2$ is Banach space.

Definition 2.5 [5]: A stochastic process (s.p.)is a collection of r.v. that is, for each $t \in T = [t_0, \infty)$, $U(t) = U_t$ is a r.v. defined on a probability space (Ω, F, P).

Definition 2.6 [5]: A s.p. $U(t)$ $t \in T = [t_0, \infty)$ is called second order s.p. or (2−s.p.) if $U(t) \in L_2$, for each $t \in T = [t_0, \infty)$. The norm of $U(t)$ denoted in the usual manner by $\|U(t)\|_2$ is defined by

$$\|U_t\|_2^2 = \langle U_t, U_t \rangle = E[(U_t)^2] \tag{8}$$

Also, a s.p. $U(t), t \in T$, where $\mathbf{E}[U^4(t)] < \infty$ for all $t \in T = [t_0, \infty)$, will be called a 4− s.p. [30, 13].

The norm of $U(t)$ denoted in the usual manner by $\|U(t)\|_4$ is defined by

$$\|U(t)\|_4 = E[(U(t))^4] \tag{9}$$

Definition 2.7 [23]: The s.p. $U(t)$, $t \in T = [t_0, \infty)$ is called Gaussian s.p. if for every

$t_0, \ldots, t_n \in T$, then random vector $U_{t_0}, \ldots, U_{t_n}$ are independent Gaussian random vector.

$$U = (U_1, U_2, \ldots, U_n)^T \tag{10}$$

Definition 2.8 [23]: If U (t) is an 2 s.p. then for each, t_1, $t_2 > 0$ the two dimensional deterministic function $\Gamma_U(t_1, t_2) = \mathbf{E}[\overline{U}(t)U(t)]$ is called the correlation function associated to U (t). The correlation function $\Gamma_U(t_1, t_2)$ of an 2−s.p. U (t) always exists since

$$\left|\Gamma_U(t_1, t_2)\right| = \left|\mathbf{E}\left[U_{t_1}U_{t2}\right]\right| \le \mathbf{E}\left|\left[U_{t_1}U_{t2}\right]\right| \le \left\|U_{t_1}\right\|_2 \left\|U_{t_2}\right\|_2 < \infty \tag{11}$$

For more details about the basic theories and properties related to m.s calculus can you see [28]. In 2019 , Sara M. Attieh used the the m.s. limit to extend the conformable fractional calculus to the random framework[4]. This section contain summarize some basic important concepts depends on m.s. mean square conformable fractional calculus .

Definition 2.9 [4]: Let U (t) ∈ T be 2− s.p., then for all t ∈ T , the mean square conformable fractional derivative (m.s.CFD) of U (t) of order $0 < \alpha \le 1$, $T^\alpha U(t)$, is defined as:

$$T_t^\alpha U_t = \underset{h\to 0}{\mathrm{l.i.m.}}\left[\frac{U(t+ht^{1-\alpha}) - U(t)}{h}\right] \tag{12}$$

that is

$$\lim_{h\to 0}\left\|\frac{U(t+ht^{1-\alpha}) - U(t)}{h} - T_t^\alpha U(t)\right\|_2 = 0 \tag{13}$$

Definition 2.10: Let U (t) ∈ T be 2− s.p., then for all t ∈ T , the partial m.s.CFD ($T^\alpha U$ (t, x)) of U_t of order $0 < \alpha \le 1$ is defined as:

$$T_t^\alpha U(t,x) = \frac{\partial^\alpha U}{\partial t^\alpha}(t,x) = \underset{h\to 0}{\mathrm{l.i.m.}}\left[\frac{U(x, t^{1-\alpha} + h) - U(x,t)}{h}\right] \tag{14}$$

Definition 2.11 [4]: Given U_t, $t \in [a,b]$ is 2−s.p., then the m.s.CFI ($I_\alpha^a U_t$) of U_t of order $0 < \alpha \le 1$ is defined by

$$I_\alpha^a U_t = \int_a^t U_s ds^\alpha = \int_a^t s^{\alpha-1} U_s ds, \quad a < t \tag{15}$$

Where the integral is the m.s. conformable fractional Riemann integral.

3. Random Fourier Transform:

It is well known fact, the s.p. is r.v. for instant of time, so the s.p. is discontinuous function. Therefore deterministic Fourier transform can not be applied for the stochastic process. This reason motivate M.-C. Casaban et al. to introduce the random Fourier transform as follows [7]:

Definition 3.1: For any m.s. of s.p., U(x) which is defined for all $x \in R$, and

Satisfy

$$\int_{-\infty}^{\infty} \| U(x) \|_2 \, dx < \infty \tag{16}$$

The random Fourier transform of the stochastic process U (x) is given by the following m.s. integral

$$\bar{U}(\Upsilon) = \Im[U(x)](\Upsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Upsilon\tau}\, U(\tau)\, d\tau, \forall \Upsilon \in R \tag{17}$$

Remark 3.1

1. The integrations in (14) is usual integral while in (16) is m.s. integral.
2. The norm in (14) is m.s. norm.
3. Any stochastic process satisfy (14) is called m.s. absolutely integrable on R.
4. Note that, $\bar{U}(\Upsilon)$ is s.p.

Example 3.1 [7]: If B is a r.v. such that $E[|B|^s] \le \nu \mu^s, \forall s \ge s_\circ$ for some non-negative integer s_0 and for some positive constants ν and μ. Then the random Fourier transform for e^{-Bx^2} is given by the stochastic process $\frac{1}{\sqrt{2B}} e^{-\frac{\Upsilon^2}{4B}}, \forall \Upsilon \in R$.

Lemma 3.1: If U (x) is a 2-s.p. such that the following properties are hold for all R:

1. The stochastic process U (x) is m.s. absolutely integrable and m.s. continuous differentiable.
2. The first and the second m.s. derivative of the stochastic process U (x) are mean square absolutely integrable and m.s. continuous differentiable.

then

$$\Im[U'(x)](\Upsilon) = i\xi\bar{U}(\Upsilon) \quad (18)$$

$$\Im[U''(x)](\Upsilon) = -\Upsilon^2\,\bar{U}(\Upsilon) \quad (19)$$

The convolution theorem plays a fundamental role in applied mathematics. In fact, the convolution is an integral operation on two given functions that creates a new function, clarifying how the shape of the first function is modified by the second function. This operation is given by the integration of the product of these two functions after the first function is reversed and shifted. In m.s. sense, the convolution theorem can be defined for any two 4-s.p. V (x) and U (x) such that

$$\int_{-\infty}^{\infty}[\| U(\tau)\|_4]^2\,d\tau<\infty, \int_{-\infty}^{\infty}[\| V(\tau)\|_4]^2\,d\tau<\infty, \quad (20)$$

Then for all $x\in(-\infty,\infty)$ the convolution s.p. of $V(x)$ and $U(x)$ is given by the following m.s. integral [7]

$$(V * U)(x)=\int_{-\infty}^{\infty} U(\tau)V(x-\tau)\,d\tau \quad (21)$$

4. Random Fractional Laplace Transform (RFLT)

Definition 4.1: The 2-s.p. U (t) belongs to the class C_α for some $\alpha \in (0, 1]$, if and only if the followingstatements are holds:

1. $U\left((\alpha t)^{\frac{1}{\alpha}}\right)$ is m.s. integrable.

2. $U\left(t\right)=0, \forall t < 0.$

3. There exist a positive constant (a) and (a) positive nonzero constant M such that

$$\| U((\alpha t)^{\frac{1}{\alpha}} \|\leq M\,e^{at} \quad (22)$$

Definition 4.2: For specific α ∈ (0, 1], if U (t) is a 2-s.p. belong to C_{α} then the random fractional Laplace transform of U (t) is define by

$$\bar{U}_{\alpha}(s) = L_{\alpha}[U(t)] = \int_{0}^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}} U(t)\, d^{\alpha}t, s \in \mathbb{C}, \mathrm{Re}(s) > a. \tag{23}$$

Remark 4.1:

1. The integral in (23) is m.s. fractional integral.

2. The norm in (22) is m.s. norm.

3. Note that, $\bar{U}(s)$ is a s.p.

Lemma 4.1: The relationship between the random Laplace transform and the RFLT is given by [17]:

$$L_{\alpha}[U(t)] = L[U((\alpha t)^{\frac{1}{\alpha}})] \tag{24}$$

Where

$$L[U(t)] = \int_{0}^{\infty} e^{-st} U(t) dt$$ is the random Laplace transform.

Proof.

$$L_{\alpha}[U(t)] = \int_{0}^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}} U(t)\, d^{\alpha}t = \int_{0}^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}} U(t)\, t^{\alpha-1} dt \tag{25}$$

Let $\upsilon = \frac{t^{\alpha}}{\alpha}$, we have $d\upsilon = t^{\alpha-1} dt$. Substitution this change of variable in equation (25), one can have

$$L_{\alpha}[U(t)] = \int_{0}^{\infty} e^{-su} U((\alpha\upsilon)^{\frac{1}{\alpha}}) d\upsilon = L[U((\alpha t)^{\frac{1}{\alpha}}] \tag{26}$$

Theorem 4.2: (Linearity): Let U (t) and V (t) are 2-s.p. U (t) is belong to the class C_α for some $\alpha \in (0, 1]$, where A, B are random variables then:

$$L_\alpha[AU(t)+BV(t)] = AL_\alpha[U(t)] + BL_\alpha[V(t)] \tag{27}$$

Proof.

$$L_\alpha[AU(t)+BV(t)] = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}[AU(t)+BV(t)]d^\alpha t$$

$$= \int_0^\infty AU(t)e^{-s\frac{t^\alpha}{\alpha}}\, d^\alpha t + \int_0^\infty BV(t)e^{-s\frac{t^\alpha}{\alpha}}\, d^\alpha t$$

$$= A\int_0^\infty U(t)e^{-s\frac{t^\alpha}{\alpha}}\, d^\alpha t + B\int_0^\infty V(t)e^{-s\frac{t^\alpha}{\alpha}}\, d^\alpha t$$

$$= AL_\alpha[U(t)] + BL_\alpha[V(t)]$$

Theorem 4.3: If the 2-s.p. U(t) is belong to class C_α for some $\alpha \in (0, 1]$,where A is random variable then :

$$L_\alpha[A] = \frac{A}{s} \tag{28}$$

Proof. $L_\alpha[A] = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} A\, d^\alpha t$

$$= A\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \frac{A}{s}$$

Theorem 4.4: If 2-s.p. U(t) is belong to class C_α for some $\alpha \in (0, 1]$, where A is random variable then :

$$L_\alpha[e^{A\frac{t^\alpha}{\alpha}}] = \frac{1}{s-A} \tag{29}$$

Proof .

$$L_\alpha[e^{A\frac{t^\alpha}{\alpha}}] = \int_0^\infty e^{A\frac{t^\alpha}{\alpha}} e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \int_0^\infty e^{(A-s)\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \frac{1}{(s-A)}$$

Theorem 4.5: Let the 2-s.p. $U(t,A) = \sin(A\frac{t^\alpha}{\alpha})\delta(t)$ and $V(t,A) = \cos(A\frac{t^\alpha}{\alpha})\delta(t)$ are belongs to class C_α for some $\alpha \in (0,1]$, where A is random variable and $\delta(t)$ is the unite step function then :

1. $L_\alpha[U(t, A)] = \frac{A}{s^2 + A^2}$. (30)

2. $L_\alpha[V(t, A)] = \frac{s}{s^2 + A^2}$. (31)

Proof.

Since, $\sin(A\frac{t^\alpha}{\alpha}) = \mathrm{Im}(e^{iA\frac{t^\alpha}{\alpha}})$, where Im denotes to the imaginary part of the complex random variable.

Now, by using simple computations, one can have

$$L_\alpha[\sin(A\frac{t^\alpha}{\alpha})\delta(t)] = \int_0^\infty \mathrm{Im}(e^{iA\frac{t^\alpha}{\alpha}})\delta(t) e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \int_0^\infty \mathrm{Im}(e^{iA\frac{t^\alpha}{\alpha}} e^{-s\frac{t^\alpha}{\alpha}}) d^\alpha t$$

$$= \mathrm{Im}\int_0^\infty e^{(iA-s)\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \mathrm{Im}(\lim_{t\to\infty}(\frac{e^{(iA-s)\frac{t^\alpha}{\alpha}}}{iA-s}) - \frac{1}{iA-s})$$

$$= \mathrm{Im}(\frac{1}{s-iA})$$

$$= \mathrm{Im}[\frac{s+iA}{(s-iA)(s+iA)}]$$

$$= \mathrm{Im}(\frac{s+iA}{s^2+A^2})$$

$$= \frac{A}{s^2+A^2}.$$

Since $\cos(A\frac{t^\alpha}{\alpha}) = \mathrm{Re}(e^{iA\frac{t^\alpha}{\alpha}})$, we can prove $L_\alpha[\cos(A\frac{t^\alpha}{\alpha})] = \frac{s}{s^2+a}$ in similar manner.

Theorem 4.2: Let the second order stochastic process U(t, x) is belongs to the class C_α for some $\alpha \in (0,1]$, and if U(t, x) is m.s. differentiable on T when variable t ∈ T , then the (RFLT) of $T_t^\alpha U(t, x)$ of order $\alpha \in (0,1]$ is defined as:

$$L_\alpha[T_t^\alpha U(t,x)] = s\bar{U}_\alpha(s,x) - U(0,x) \tag{32}$$

Proof.

$$L_\alpha[T_t^\alpha U(t,x)] = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} T_t^\alpha U(t,x)\, d^\alpha t$$

$$= e^{-s\frac{t^\alpha}{\alpha}} U(t,x)|_0^\infty + s\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} U(t,x)\, d^\alpha t$$

$$= s\bar{U}_\alpha(s,x) - U(0,x).$$

Theorem 4.3: Let the s.p. U(t, x) is belongs to the class C_α for some α ∈ (0, 1], and if U(t, x) is m.s. differentiable on T when variable t ∈ T , then the (RFLT) of $\frac{\partial U(t, x)}{\partial t}$ of order $\alpha \in (0,1]$ is defined as:

$$1.\ L_\alpha[\frac{\partial U(t,x)}{\partial x}] = \frac{d\bar{U}_\alpha(s,x)}{dx} \tag{33}$$

$$2.\ L_\alpha[\frac{\partial^2 U(t,x)}{\partial x^2}] = \frac{d^2\bar{U}_\alpha(s,x)}{dx^2} \tag{34}$$

Proof.(1)

$$\frac{d\bar{U}_\alpha(t,x)}{dx} = \frac{d}{dx}\int_0^\infty U(t,x)\,e^{-s\frac{t^\alpha}{\alpha}}d^\alpha t$$

$$= \int_0^\infty \frac{\partial U(t,x)}{\partial x} e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= L_\alpha[\frac{\partial U(t,x)}{\partial x}].$$

We can use the same manner of proof (1) to prove (2).

5. Solving Random Time-Fractional Heat Equation

This section devoted to solving the random time-fractional heat diffusion (conduction) in one dimensional infinite medium. The proposed procedure is suggested applying the RFLT together with applying the RFT with respect to t and x respectively.

We start the analysis by the following lemma which is explain the ability of exchange between the random fractional Laplace operator and random Fourier operator.

Lemma 5.1: Let $\Xi = \{(t,x)\,|\,x \in R,\ \ t \geq 0\}$, if $U(t,x)$ be a 2-s.p. $\forall$ (t, x) $\in \Xi$ such that:

1. $U(t,x)$ is m.s. time-fractional integrable.

2. $\int_{-\infty}^{\infty} \| U(t,x) \|_2 \, dx < \infty, \forall\, t \geq 0$

3. If there exist c positive constant and (c) positive nonzero constant M such that :

$$\| U(t,x) \|_2 \leq Me^{ct}, \forall\, x \in R.$$

Then

$$\Im(L_\alpha(U(t,x))) = L_\alpha(\Im(U(t,x))) \tag{35}$$

Where L_α is the RFLT and $\Im$ is the random Fourier transform with respect to t and x respectively.

Proof. By using the definitions of the random fractional Laplace operators and the random Fourier operators and the exchange of integrations order , we can easily prove as follows :

$$L_\alpha(\Im(U(t,x))) = \int_0^\infty \Im(U(t,x) e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{-\infty}^\infty U(t,x) e^{-i\Upsilon x} e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha-1} \, dx \, dt \tag{36}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty U(t,x) e^{-i\Upsilon x} e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha-1} \, dt \, dx \tag{37}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty U(t,x) e^{-i\Upsilon x} e^{-s\frac{t^\alpha}{\alpha}} d^\alpha t \, dx \tag{38}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty L_\alpha(U(t,x)) e^{-i\Upsilon x} \, dx \tag{39}$$

$$= \Im(L_\alpha(U(t,x))).$$

Theorem 5.1: The analytic solution of the random time-fractional heat diffusion (conduction) in one dimensional infinite medium which governed by (1) - (2) is

$$U(t,x) = \sqrt{\frac{\alpha}{2At^\alpha}} \int_{-\infty}^\infty \Psi(\tau - x, B) e^{\frac{-\alpha\tau^2}{4At^\alpha}} d\tau \tag{40}$$

Proof. By taking the random fractional Laplace transform with respect to t to both sides of (1), it follows :

$$L_\alpha(T_t^\alpha U(t,x)) = L_\alpha(A \frac{\partial^2 U}{\partial x^2}(t,x)) \tag{41}$$

By using the properties of RFLT which stated in chapter three , it follows:

$$s\bar{U}_\alpha(s,x) - U(0,x) = A \frac{\partial^2 \bar{U}_\alpha}{dx^2}(s,x) \tag{42}$$

By using the initial condition(2), one can have

$$s\bar{U}_\alpha(s,x) - \Psi(x,B) = A \frac{d^2 \bar{U}_\alpha}{dx^2}(s,x) \tag{43}$$

In order to solve (43) we apply the Fourier transform with respect to x as follows:

$$\Im(s\bar{U}_{\alpha}(s,x)-\Psi(x,B))=\Im(A\frac{d^2\bar{U}_{\alpha}}{dx^2}(s,x)) \quad (44)$$

By using lemma (3.1), implies to:

$$s\Im(\bar{U}_{\alpha}(s,x))-\Im(\Psi(x,B))=-A\Upsilon^2\Im(\bar{U}_{\alpha}(s,x)) \quad (45)$$

By using lemma (5.1), we have

$$sL_{\alpha}(\Im(U(t,x)-\Im(\Psi(x,B))=-A\Upsilon^2 L_{\alpha}(\Im U(t,x)) \quad (46)$$

$$L_{\alpha}(\Im(U(t,x))=\frac{\Im(\Psi(x,B))}{A\Upsilon^2+s} \quad (47)$$

Applying the inverse of the random fractional Laplace transform for (47) as follows :

$$\Im(U(t,x))=\Im(\Psi(x,B))e^{-A\Upsilon^2\frac{t^{\alpha}}{\alpha}} \quad (48)$$

By using the result in example (3.1),we have

$$\sqrt{\frac{\alpha}{2At^{\alpha}}}\,\Im(e^{-\frac{\alpha x^2}{4At^{\alpha}}})=e^{-\Upsilon^2\frac{t^{\alpha}}{\alpha}} \quad (49)$$

By substituting (39) in (38), we get

$$\Im(U(t,x))=\sqrt{\frac{\alpha}{2At^{\alpha}}}\,\Im(\Psi(x,B))\,\Im(e^{-\frac{\alpha x^2}{4At^{\alpha}}}) \quad (50)$$

Applying the inverse of the random Fourier transform and the convolution theorem , the analytic solution will take the form:

$$U(t,x)=\sqrt{\frac{\alpha}{2At^{\alpha}}}\int_{-\infty}^{\infty}\Psi(\tau-x,B)\frac{e^{-\frac{\alpha x^2}{4At^{\alpha}}}}{\sqrt{A}}d\tau \quad (51)$$

Corollary 5.1: under all the assumption related to the r.v.s. K and A which are stated in this paper, the first, $E[U(t,x)]$, and second moment, $E[U^2(t,x)]$, are given by

$$E[U(t,x)]=\sqrt{\frac{\alpha}{2t^{\alpha}}}\int_{-\infty}^{\infty}E[\Psi(\tau-x,B)]E[\frac{e^{-\frac{\alpha\tau^2}{4At^{\alpha}}}}{\sqrt{A}}]d\tau \quad (52)$$

$$E[U^2(t,x)]=\frac{\alpha}{2t^{\alpha}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E[\Psi(\tau_1-x,B)\Psi(\tau_2-x,B)]\times E[\frac{e^{-\frac{\alpha(\tau_1^2+\tau_2^2)}{4At^{\alpha}}}}{A}]d\tau_1\,d\tau_2 \quad (53)$$

Proof . By taking the expectation to the both sides of the s.p. in equation(40), one can have

$$E[U(t,x)]=E[\sqrt{\frac{\alpha}{2t^{\alpha}}}\int_{-\infty}^{\infty}\Psi(\tau-x,B)\frac{e^{-\frac{\alpha\tau^2}{4At^{\alpha}}}}{\sqrt{A}}d\tau] \quad (54)$$

By using the property of switching the integration and expectation, the above equation become in the following form

$$E[U(t,x)]=\sqrt{\frac{\alpha}{2t^{\alpha}}}\int_{-\infty}^{\infty}E[\Psi(\tau-x,B)\frac{e^{-\frac{\alpha\tau^{2}}{4At^{\alpha}}}}{\sqrt{A}}]d\tau \tag{55}$$

Since the r.v.s. A and B are independent, we can rewrite equation(55) as follows

$$E[U(t,x)]=\sqrt{\frac{\alpha}{2t^{\alpha}}}\int_{-\infty}^{\infty}E[\Psi(\tau-x,B)E[\frac{e^{-\frac{\alpha\tau^{2}}{4At^{\alpha}}}}{\sqrt{A}}]d\tau \tag{56}$$

To prove eq (53) , we square the both sides of the equation (40)

$$U^{2}(t,x)=\frac{\alpha}{2t^{\alpha}}\int_{-\infty}^{\infty}\Psi(\tau_{1}-x,B)\Psi(\tau_{2}-x,B)\frac{e^{-\frac{\alpha(\tau_{1}^{2}+\tau_{2}^{2})}{4At^{\alpha}}}}{A}]d\tau_{1}d\tau_{2} \tag{57}$$

Now, by taking the expectation to both sides of stochastic process , one can have

$$E[U^{2}(t,x)]=\frac{\alpha}{2t^{\alpha}}E[\int_{-\infty}^{\infty}\Psi(\tau_{1}-x,B)\Psi(\tau_{2}-x,B)\frac{e^{-\frac{\alpha(\tau_{1}^{2}+\tau_{2}^{2})}{4At^{\alpha}}}}{A}d\tau_{1}d\tau_{2}] \tag{58}$$

By entering the expectation into the integration of equation, we have

$$E[U^{2}(t,x)]=\frac{\alpha}{2t^{\alpha}}\int_{-\infty}^{\infty}E[\Psi(\tau_{1}-x,B)\Psi(\tau_{2}-x,B)\frac{e^{-\frac{\alpha(\tau_{1}^{2}+\tau_{2}^{2})}{4At^{\alpha}}}}{A}]d\tau_{1}d\tau_{2} \tag{59}$$

Since the r.v.s. A and B are independent, we can rewrite the equation as follows

$$E[U^{2}(t,x)]=\frac{\alpha}{2t^{\alpha}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E[\Psi(\tau_{1}-x,B)\Psi(\tau_{2}-x,B)]\times E[\frac{e^{-\frac{\alpha(\tau_{1}^{2}+\tau_{2}^{2})}{4At^{\alpha}}}}{A}]d\tau_{1}d\tau_{2} \tag{60}$$

Illustrative example

This section is devoted to giving a numerical example to explain the theoretical results which are introduced in the previous sections. In fact, we will focus on computing the mean and the variance of the stochastic process solution for different value of alpha and for different value of t. The Maple (16) computer software has been used to carry out this task.

Example 6.1: Consider the random time-fractional heat diffusion equation in an infinite medium (1)-(2) when the medium is an inhomogeneous material so that it is better to represent to thermal diffusion coefficient by random variables. Where the thermal diffusion coefficient A be a fourth order random variable which has beta distribution B(2, 1) and satisfy all related conditions which is stated in this paper. Also, we assume that the initial condition is $\Psi(x, B) = e^{-x^2}$. In fact this example has been studied when $\alpha = 1$ by M.-C. Casabán el al. [6]. In order to compute $E[\frac{e^{-\frac{\alpha\tau^2}{4At^\alpha}}}{\sqrt{A}}]$ and $E[\frac{e^{-\frac{\alpha(\tau_1^2+\tau_2^2)}{4At^\alpha}}}{\sqrt{A}}]$. As in [6] the integration in equation (34) and (35) has been approximated to

$$E[U(t,x)] = \sqrt{\frac{\alpha}{2t^\alpha}} \int_{-M}^{M} E[\Psi(\tau - x, B)] E[\frac{e^{-\frac{\alpha\tau^2}{4At^\alpha}}}{\sqrt{A}}] d\tau \tag{61}$$

$$E[U^2(t,x)] = \frac{\alpha}{2t^\alpha} \int_{-M}^{M} \int_{-M}^{M} E[\Psi(\tau_1 - x, B)\Psi(\tau_2 - x, B)] \times E[\frac{e^{-\frac{\alpha(\tau_1^2+\tau_2^2)}{4At^\alpha}}}{A}] d\tau_1 \, d\tau_2 \tag{62}$$

where M = 4. The Newton-Cotes method has been implemented to find the approximate integral in equations (40) and (41). The results can be summarized in the following figures. Where Figure(1)-Figure(5) represent the approximate value of $\mathbf{E}[U(t,x)]$ for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 when t = 0.1, 1, 10, 50, 100. While Figure(6) - Figure(10) represent approximate value of the standard deviation of $U(t,x)$ for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 when t = 0.1, 1, 10, 50, 100. The results of this example are agreed with the results in [6] of when α = 1.

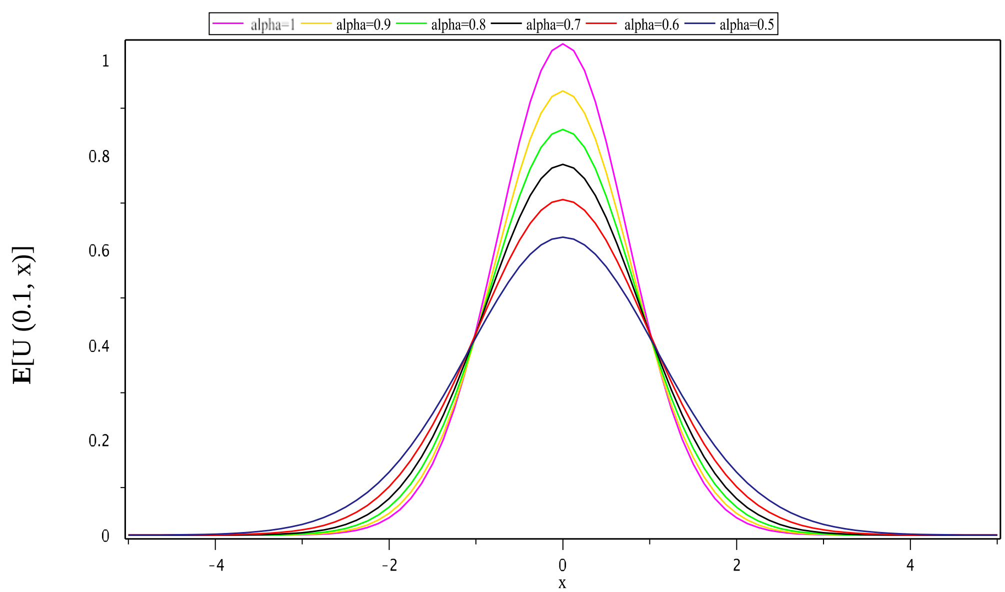

Figure 1: The approximate for expectation **E**[U (t, x)], for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t =[0, 1] .

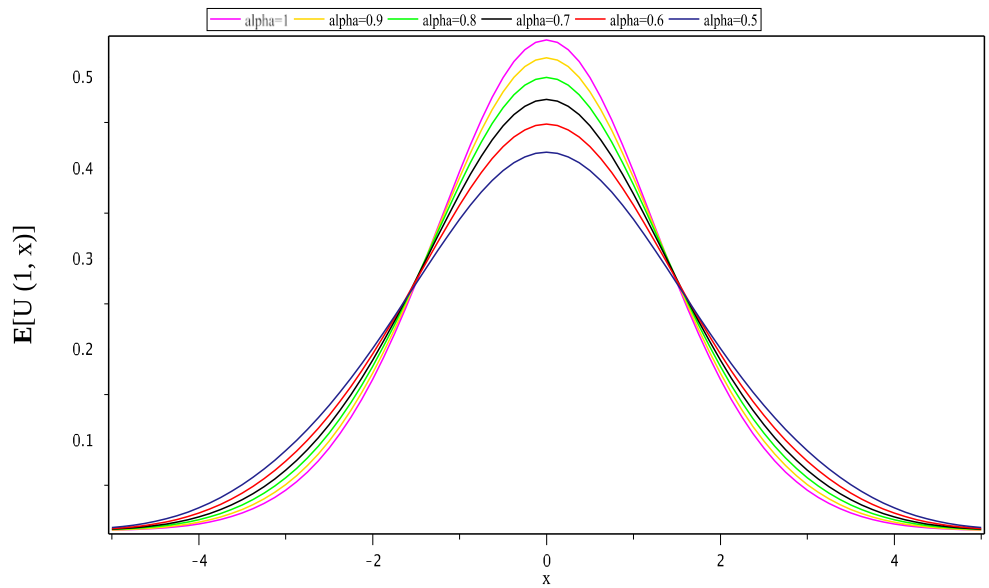

Figure 2: The approximate for expectation **E**[U (t, x)], for deferent value of α = 0.5, 0.6, 0.7, 0.8, .9, 1 at t= [0, 0.5] .

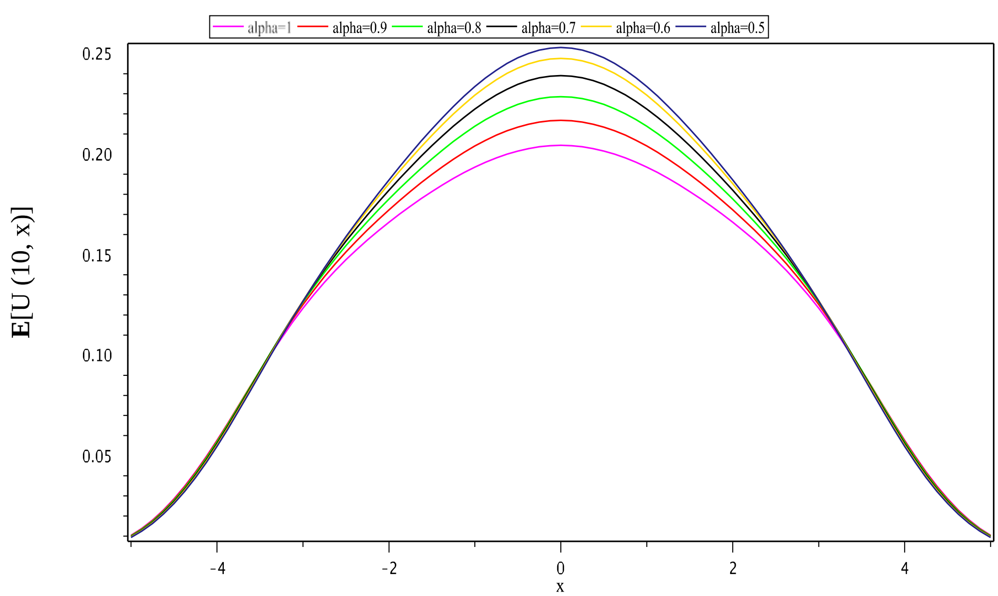

Figure 3: The approximate for expectation **E**[U (t, x)], for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t = [0, 0.25] .

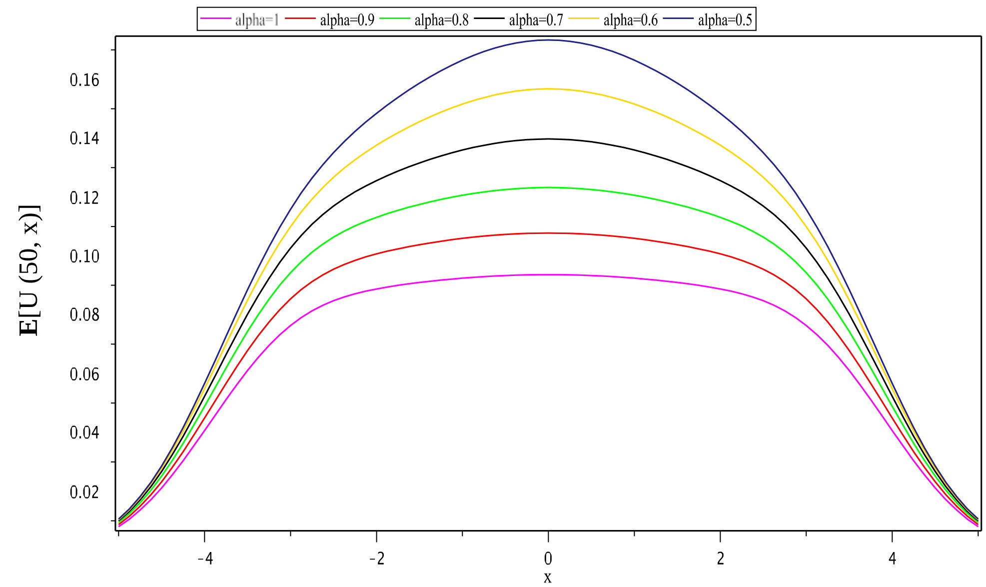

Figure 4: The approximate for expectation **E**[U (t, x)], for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t= [0, 0.16].

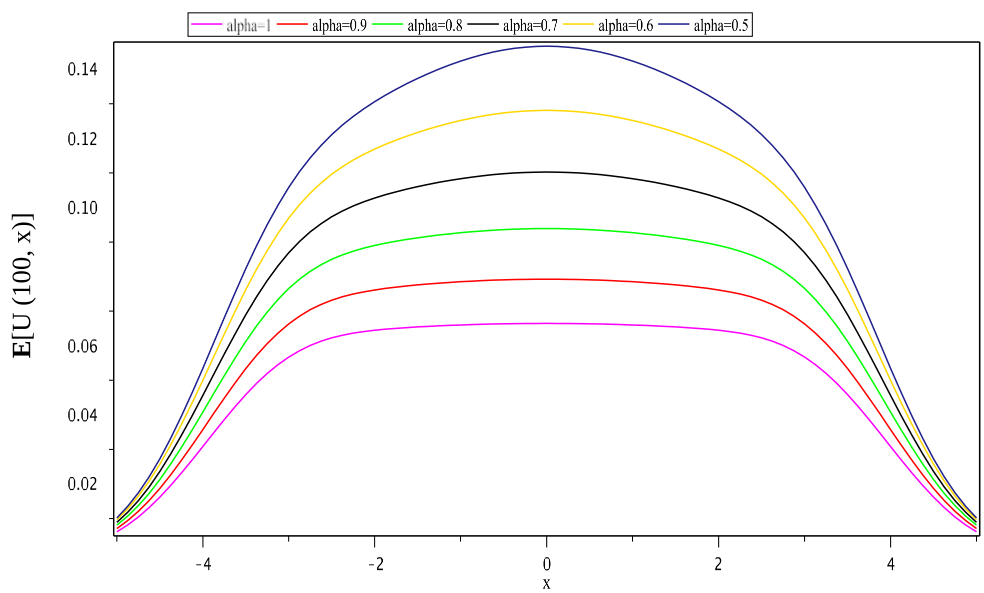

Figure 5: The approximate for expectation **E**[U (t, x)], for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t= [0, 0.14] .

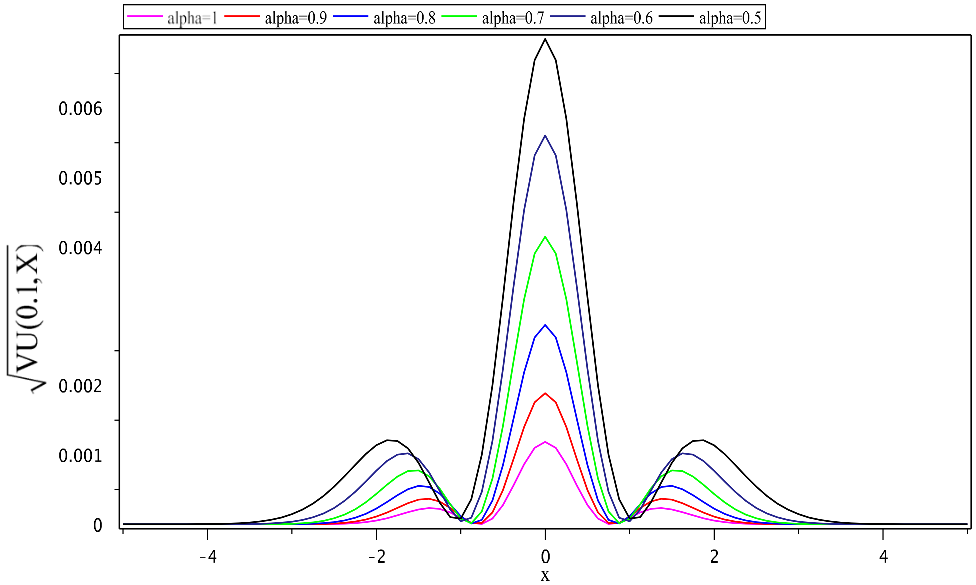

Figure 6: The approximate the standard deviation of U (t, x), for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t= [0, 0. 006].

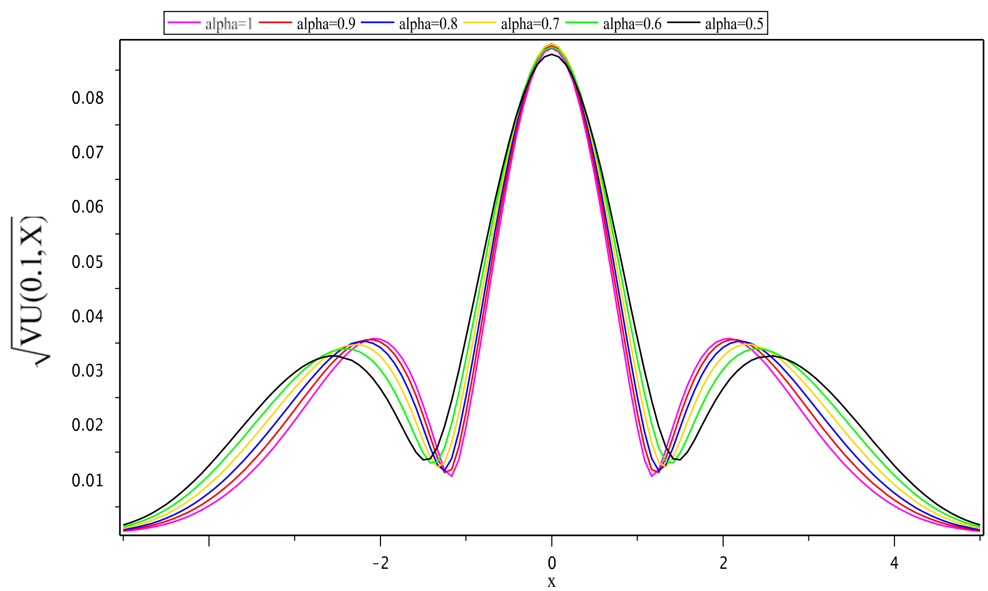

Figure 7: The approximate the standard deviation of U (t, x), for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t= [0, 0.08].

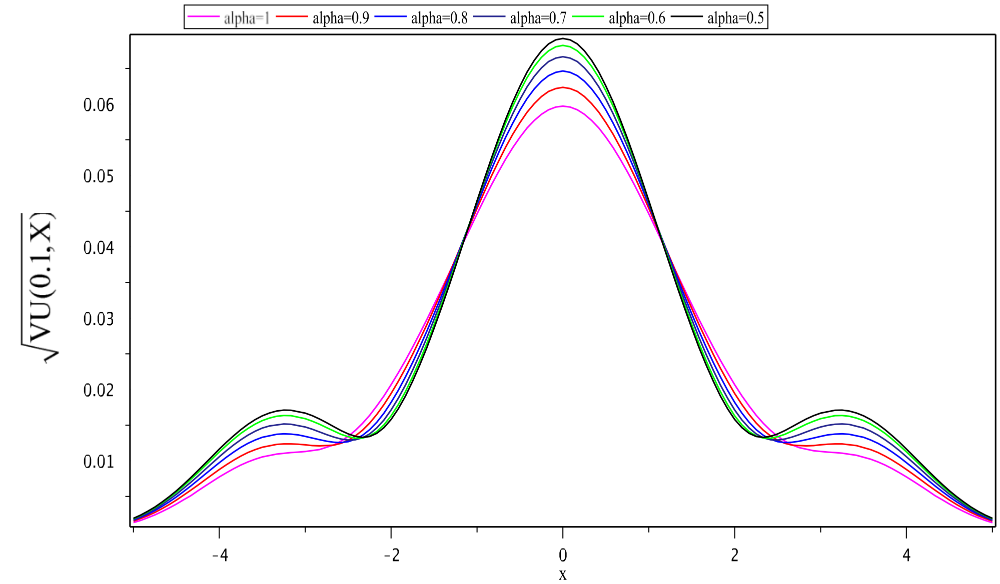

Figure 8: The approximate the standard deviation of U (t, x), for deferent value of $\alpha = 0.5$, 0.6, 0.7, 0.8, 0.9, 1 at t= [0, 0.6].

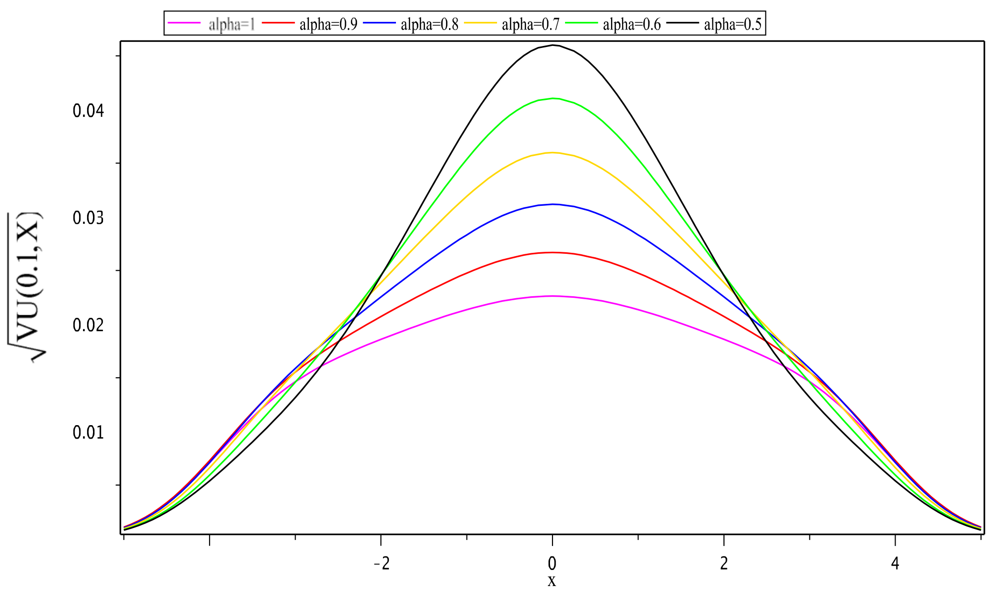

Figure 9: The approximate the standard deviation of U (t, x), for deferent value of $\alpha = 0.5$, 0.6, 0.7, 0.8, 0.9, 1 at t=[0,0.4]

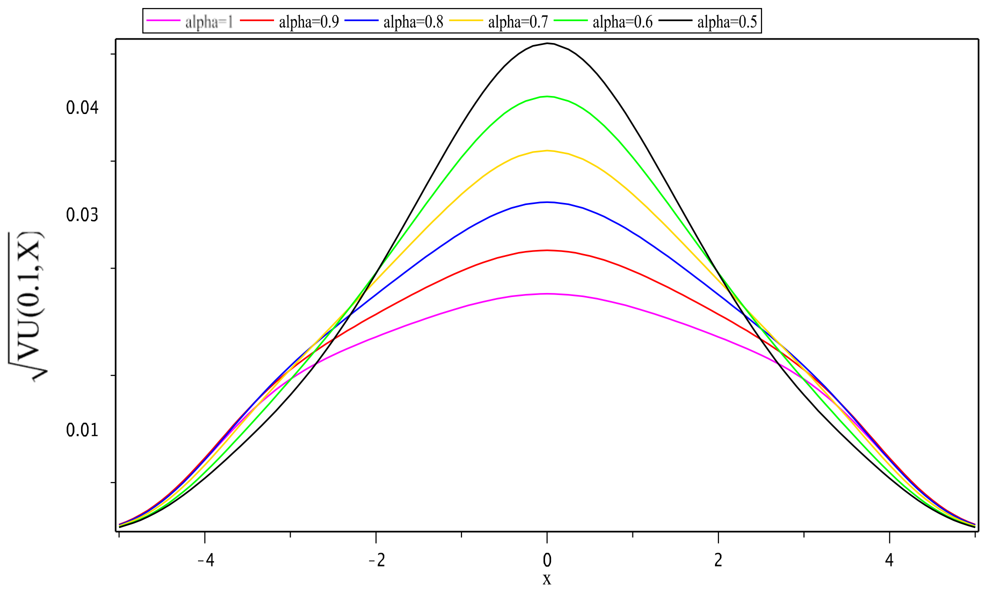

Figure 10: The approximate the standard deviation of U (t, x), for deferent value of α = 0.5, 0.6, 0.7, 0.8, 0.9, 1 at t=[0,0.4]

6. Conclusions

It is well known fact that the heat diffusion equation investigates the concept of spread of heat randomly in the medium at different rates and in any directions. The physical principle of the conductivity constant is associated with the speed of the heat flux of the medium during the temperature changing over time. The spread rate of the heat is proportional to the conductivity constant. Therefore the conductivity constant plays a fundamental role in the behavior of the solution of the heat diffusion equation. If the medium is an inhomogeneous material (containing impurities), then it is better to represent to conductivity constant by random variables. The other hand, the implementation of fractional derivatives instead of the classical derivatives to describe the heat diffusion equation is more nature since the fractional derivatives describes the changes in the non-local region, while the classical derivatives are in local. In this paper the solution of random time-fractional heat diffusion equation has been investigated in an infinite medium. Through this work, we interested in applying the RFLT together with random Fourier transform to solve random time-fractional heat diffusion equation has been investigated in an infinite medium and study, the mean and variance of the stochastic process solution for different values of fractional derivative order.

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تحويل لابلاس الكسوري العشوائي لحل مسائل الحرارة الكسورية العشوائية المرتبطة بالزمن في وسط غير منته

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المستخلص

تعد التحويلات التكاملية العشوائية من الادوات المهمة لحل مسائل انتشار الحرارة العشوائية الا انه لايمكن تطبيقها لايجاد حل معادلات الحرارة الكسورية العشوابية المرتبطة بالزمن عندما يكون مربع التوقع عبارة عن مربع توقع كسوري يعتمد صيغة مشتقة المتطابقة الكسورية المرتبطة بالزمن . هذه الدراسة تقدم تعميما لتحويل لابلاس العشوائي الى تحويل لابلاس الكسوري العشوائي بأستخدام مشتق كسوري بصيغة المشتقة التطابقي لحل مسائل الحرارة الكسورية العشوائية المرتبطة بالزمن .حيث يتم ايجاد حل للمسألة اعلاه بستخدام صيغتي تحويل لابلاس الكسوري العشوائي مع تحويل فوريير العشوائي , ثم حساب التوقع والتباين بقيم كسورية مختلفة .