Beta-Hyperhalfnormal Distribution and Its Application

Maradesa Adeleke

Federal University of Technology Akure, Nigeria

Corresponding author: e-mail: maradesasta121378@futa.edu.ng

Doi 10.29072/basjs.202021

Abstract

This research developed Hyper halfnormal distribution (HHND) and Beta-hyper halfnormal distribution (BHHND). The statistical properties of those distributions were studied and BHHND is found to have bathtube hazard function. The distributions are fitted to lifetime data that seemed to have bathtube distribution. From the analysis, the definition of HHND depends on the value of p and q (mixing proportion) and coefficient of variation. For q > p, we have Hyper halfnormal and it is Hypo-halfnormal if otherwise. The BHHND is useful in modeling heavily skewed, non-normal data with bathtube hazard function.

Article inf.

Received:
30/9/2019

Accepted

20/5/2020

Published

31/8/2020

Keywords:
Hyper-halfnormal distribution (HHND), Beta-hyper halfnormal distribution (BHHND), Halfnormal distribution (HND), Moment generating function, Bathtub Hazard Function

1 Introduction

In statistics and probability, the statistical properties and interrelationship between random variables play cogent roles in defining, explaining and modeling naturally many real life processes. In essence, numerous probability distributions have been proposed and defined in the literature, which are found to be applicable and mathematically tractable in modeling many real life complex processes or to explain many events in real life. Various methods exist for defining statistical distributions. Many of these have risen from the need to model naturally occurring phenomena or events. For example, the normal distribution explains real-valued variables that tend to cluster around a single mean value, whereas the Poisson distribution models discrete rare events, Pogany & Nadarajah[1]. Some hyper distribution has been proposed such as hyperexponential and hyper-erlang distribution; Orlik & Rappaport[2] proposed hyper-erlang distribution and it was applied in modeling the channel holding time and cell resistant time in mobile computing by displaying that coefficient of variation could be varied to be less than or greater than one. To introduce the pattern of skewness into probability density function (PDF) of any baseline distribution in order to improve its flexibility in capturing data with heavily tailed and highly skewed distribution, the beta distribution, as among the skewed distributions, which rescale parent distributions in order to create series of shapes. Beta-generated family of distribution has been use to create multiplicities of convoluted beta family of distribution with increased scope in modeling non-normal data. Akomolafe & Maradesa[3] developed Beta-Halfnormal Distribution by using Logit of Beta function defined by Jones[4], the statistical properties of this distribution adequately studies. Akinsete, Famoye & Lee[5] developed Beta-pareto and the distribution was applied to flood data set which depicted that it is unimodal and provides a good fit to the data. Eugene, Lee, & Famoye[6] also proposed Beta-normal distribution and studied its statistical properties. Idowu, Ikegwu & Emmanuel[7] derived Beta-weighted weibull and Miroslav and Nadarajah[8] developed new lifetime distribution, its statistical properties were derived and comparison is made with some selected lifetime distributions. In this research, the mixture model is derived using the mixing proportion that led to the derivation of Hyper halfnormal distribution. If $X \sim N(0,\sigma^2)$, then $Z = |X|$ follows Halfnormal which folds at mean of normal $(0, \sigma)$. Let $X_1 \sim N(0, \sigma^2{}_1)$ and $X_2 \sim N(0, \sigma^2{}_2)$ where X_1 and X_2 follow halfnormal distribution and we use the convolution of the random variable X_1 and X_2 to form the convoluted distribution called Hyper halfnormal distribution (HHND); the beta version of this convoluted distribution was latter obtained called Beta-hyper halfnormal distribution (BHHD) using Logit of beta function in order to improve the flexibility of HHND in modeling lifetime data. Finally the derived convoluted distribution BHHND and HHND were fitted to lifetime data (collected by Aarset[9])) and compared with Halnormal distribution using (HND) using performance criteria.

1.1 Derivation of HyperHalfNormal Distribution(HHND)

Given that $f(x;\sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad x > 0$ and $F(x;\sigma) = erf\left(\frac{x}{\sigma\sqrt{2}}\right)$ are the pdf and cdf of Halfnormal distribution respectively, then the Hyper halfnormal distribution can be viewed as:

$$f_{HHND}(x\ \sigma_i) = \sum_{i=1}^{n} p_i \frac{\sqrt{2}}{\sigma_i\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_i^2}},\ \text{where } i = 1,2,3\ldots,n \text{ and } p_1 + p_2 + ,\ldots, + p_n = 1$$

When p + q = 1, then the pdf becomes (1)

$$f_{HHND}(x\ \sigma_1,\sigma_2) = p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_2^2}}\ ;\ \sigma_1,\sigma_2 > 0 \quad x > 0 \qquad (1)$$

$$F_{HHND}(x;\ \sigma_1,\sigma_2) = perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right) \qquad (2)$$

The (1) and (2) above are the pdf and cdf of HyperHalfNormal Distribution

1.2 Moment

$$Ex^r = \int_0^\infty x^r\left[p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_2^2}}\right]dx$$

$$= \int_0^\infty x^r p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_1^2}}dx + \int_0^\infty x^r q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_2^2}}dx \qquad (3)$$

$$= p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}\int_0^\infty x^r e^{-\frac{x^2}{2\sigma_1^2}}dx + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}\int_0^\infty x^r e^{-\frac{x^2}{2\sigma_2^2}}dx \qquad (4)$$

Let $y = \frac{x^2}{2\sigma_1^2}\ ;\ dx = \frac{\sigma_1^2 dy}{x}\ \ and\ m = \frac{x^2}{2\sigma_2^2}\ ;\ dx = \frac{\sigma_2^2 dz}{x}$

$$= p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}\int_0^\infty x^r e^{-y}\frac{\sigma_1^2 dy}{x} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}\int_0^\infty x^r e^{-z}\frac{\sigma_2^2 dz}{x} \qquad (5)$$

$$= p\frac{\sigma_1{}^2\sqrt{2}}{\sigma_1\sqrt{\pi}}\ \int_0^\infty x^{r-1}\,e^{-y}dy\ +\ q\,\frac{\sigma_2{}^2\sqrt{2}}{\sigma_2\sqrt{\pi}}\int_0^\infty x^{r-1}\ e^{-z}\ dz \qquad (6)$$

$$= p\,\frac{\sigma_1\sqrt{2}}{\sqrt{\pi}}\ \int_0^\infty\left(\sigma_1\sqrt{2y}\right)^{r-1}e^{-y}dy +\ q\frac{\sigma_2\sqrt{2}}{\sqrt{\pi}}\ \int_0^\infty\left(\sigma_2\sqrt{2z}\right)^{r-1}e^{-z}\ dz \qquad (7)$$

$$= p\,\frac{\sigma_1\sqrt{2}\left(\sigma_1\sqrt{2}\right)^{r-1}}{\sqrt{\pi}}\int_0^\infty y^{r-1}e^{-y}\,dy + q\,\frac{\sigma_2\sqrt{2}\left(\sigma_2\sqrt{2}\right)^{r-1}}{\sqrt{\pi}}\ \int_0^\infty z^{r-1}\,e^{-z}dz \qquad (8)$$

$$= p\,\frac{\sigma_1\sqrt{2}\left(\sigma_1\sqrt{2}\right)^{r-1}}{\sqrt{\pi}}\,\Gamma\!\left(\frac{r+1}{2}\right) + q\,\frac{\sigma_2\sqrt{2}\left(\sigma_2\sqrt{2}\right)^{r-1}}{\sqrt{\pi}}\ \ \Gamma\!\left(\frac{r+1}{2}\right) \qquad (9)$$

$$EX^r = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^r}{\sqrt{\pi}}\,\Gamma\!\left(\frac{r+1}{2}\right) + q\,\frac{\left(\sigma_2\sqrt{2}\right)^r}{\sqrt{\pi}}\ \ \Gamma\!\left(\frac{r+1}{2}\right)\quad ; r = 1,2,3,\ldots \qquad (10)$$

$$EX\ =\ p\frac{\sigma_1\sqrt{2}}{\sqrt{\pi}}\,\Gamma(1) +\ q\frac{\sigma_2\sqrt{2}}{\sqrt{\pi}}\ \ \Gamma(1)\ =\ p\frac{\sigma_1\sqrt{2}}{\sqrt{\pi}} +\ q\frac{\sigma_2\sqrt{2}}{\sqrt{\pi}}\ =\ \sqrt{\frac{2}{\pi}}\,(p\sigma_1\ +\ q\sigma_2)\ ; r = 1$$

$$EX^2 = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^2}{\sqrt{\pi}}\,\Gamma\!\left(\frac{3}{2}\right) + q\,\frac{\left(\sigma_2\sqrt{2}\right)^2}{\sqrt{\pi}}\ \ \Gamma\!\left(\frac{3}{2}\right)\quad r\ = 2 \qquad (11)$$

$$EX^2 = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^2}{\sqrt{\pi}}\ \frac{\sqrt{\pi}}{2} + q\,\frac{\left(\sigma_2\sqrt{2}\right)^2}{\sqrt{\pi}}\ \frac{\sqrt{\pi}}{2}\quad ;\quad r\ = 2 \qquad (12)$$

$$EX^2\ = p\,\frac{\left(\sigma_1\sqrt{2}\right)^2}{2}\ +\ q\,\frac{\left(\sigma_2\sqrt{2}\right)^2}{2}\ \ =\ p\sigma_1{}^2\ +\ q\sigma_2{}^2\ ;\ r = 2$$

$$Ex^3 = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^3}{\sqrt{\pi}}\,\Gamma(2) +\ q\,\frac{\left(\sigma_2\sqrt{2}\right)^3}{\sqrt{\pi}}\ \ \Gamma(2)\qquad r\ =\ \ 3 \qquad (13)$$

$$Ex^3 = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^3}{\sqrt{\pi}} +\ q\,\frac{\left(\sigma_2\sqrt{2}\right)^3}{\sqrt{\pi}}\ \ =\ \sqrt{\frac{2}{\pi}}\,(2p\sigma_1{}^3\ +\ \ 2q\sigma_2{}^3)\ ;\ r\ = 3 \qquad (14)$$

$$Ex^4 = \ p\,\frac{\left(\sigma_1\sqrt{2}\right)^4}{\sqrt{\pi}}\,\Gamma\!\left(\frac{5}{2}\right) +\ q\,\frac{\left(\sigma_2\sqrt{2}\right)^4}{\sqrt{\pi}}\ \ \Gamma\!\left(\frac{5}{2}\right)\quad r\ \ =\ \ 4 \qquad (15)$$

$$Ex^4 = \frac{3p}{4}\,.4\sigma_1{}^4 + \frac{3q}{4}\,.4\sigma_2{}^4\ =\ 3p\sigma_1{}^4\ \ +\ 3q\sigma_2{}^4\ \ r\ \ =\ \ 4$$

Variance: $\mu_2\ \ =\ \ Ex^2\ -\ (Ex)^2$

$$\mu_2 = p\sigma_1{}^2 + q\sigma_2{}^2 - \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2$$

$$\mu_3 = E(x-\mu)^3$$

By applying Binomial Expansion

$$\mu_3 = E\left(\binom{3}{0}x^0.(-\mu)^{3-0} + \binom{3}{1}x^1.(-\mu)^{3-1} + \binom{3}{2}x^2.(-\mu)^{3-2} + \binom{3}{3}.x^3.(-\mu)^{3-3}\right)$$

$$\mu_3 = E((-\mu)^3 + 3x(-\mu)^2 + 3x^2(-\mu) + x^3) = E(x^3 - 3\mu x^2 + 3x\mu^2 - \mu^3)$$

$$\mu_3 = Ex^3 - 3\mu Ex^2 + 3\mu^2 Ex - \mu^3$$

$$\mu_3 = \sqrt{\frac{2}{\pi}}\,(2p\sigma_1{}^3 + 2q\sigma_2{}^3) - 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)(p\sigma_1{}^2 + q\sigma_2{}^2) + 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right) - \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3$$

$$\mu_3 = \sqrt{\frac{2}{\pi}}\,(2p\sigma_1{}^3 + 2q\sigma_2{}^3) - 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)(p\sigma_1{}^2 + q\sigma_2{}^2) + 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3 - \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3$$

$$\mu_3 = \sqrt{\frac{2}{\pi}}\,(2p\sigma_1{}^3 + 2q\sigma_2{}^3) - 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)(p\sigma_1{}^2 + q\sigma_2{}^2) + 2\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3$$

$$=\mu_4 = E(x-\mu)^4$$

$$= \binom{4}{0}.x^0.(-\mu)^{4-0} + \binom{4}{1}.x^1.(-\mu)^{4-1} + \binom{4}{2}x^2.(-\mu)^{4-2} + \binom{4}{3}.x^3.(-\mu)^{4-3} + \binom{4}{4}.x^4.(-\mu)^{4-4}$$

$$=E\left[(-\mu)^4 + 4x(-\mu)^3 + 6x^2(-\mu)^2 + 4x^3(-\mu) + x^4\right] =$$

$$\mu_4 = Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 4\mu^3 Ex + \mu^4$$

$$\mu_4 = 3p\sigma_1^{\,4} + 3q\sigma_2^{\,4} - 4\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)\left(\sqrt{\frac{2}{\pi}}\,(2p\sigma_1^{\,3} + 2q\sigma_2^{\,3})\right) + 6\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2 (p\sigma_1^{\,2} + q\sigma_2^{\,2}) - 4\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right) + \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^4$$

$$= 3p\sigma_1^{\,4} + 3q\sigma_2^{\,4} - 4\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)\left(\sqrt{\frac{2}{\pi}}\,(2p\sigma_1^{\,3} + 2q\sigma_2^{\,3})\right) + 6\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2 (p\sigma_1^{\,2} + q\sigma_2^{\,2}) - 4\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^4 + \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^4$$

$$\mu_4 = 3p\sigma_1^{\,4} + 3q\sigma_2^{\,4} - 4\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)\left(\sqrt{\frac{2}{\pi}}\,(2p\sigma_1^{\,3} + 2q\sigma_2^{\,3})\right) + 6\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2 (p\sigma_1^{\,2} + q\sigma_2^{\,2}) - 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^4$$

Skeweness

$$\boldsymbol{\gamma_1(x)} = \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{\left(\sqrt{\frac{2}{\pi}}\,(2p\sigma_1^{\,3} + 2q\sigma_2^{\,3}) - 3\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)(p\sigma_1^{\,2} + q\sigma_2^{\,2}) + 2\left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^3\right)^2}{\left(p\sigma_1^{\,2} + q\sigma_2^{\,2} - \left(\sqrt{\frac{2}{\pi}}\,(p\sigma_1 + q\sigma_2)\right)^2\right)^3}$$

Kurtosis

$$\gamma_2(x)=\frac{\mu_4}{(\mu_2)^2} = \frac{\begin{pmatrix} 3p\sigma_1^4 + 3q\sigma_2^4 - 4\left(\sqrt{\frac{2}{\pi}}(p\sigma_1 + q\sigma_2)\right)\left(\sqrt{\frac{2}{\pi}}(2p\sigma_1^3 + 2q\sigma_2^3)\right) + \\ 6\left(\sqrt{\frac{2}{\pi}}(p\sigma_1 + q\sigma_2)\right)^2 (p\sigma_1^2 + q\sigma_2^2) - 3\left(\sqrt{\frac{2}{\pi}}(p\sigma_1 + q\sigma_2)\right)^4 \end{pmatrix}}{\left(p\sigma_1^2 + q\sigma_2^2 - \left(\sqrt{\frac{2}{\pi}}(p\sigma_1 + q\sigma_2)\right)^2\right)^2}$$

1.3 Reliability

$$\mathbf{R(t)} = 1 - F_{HHND}(x;\ \sigma_1, \sigma_2)$$

$$F_{HHND}(x;\ \sigma_1, \sigma_2) = perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)$$

$$= 1 - \left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)$$

1.4 Hazard Rate Function

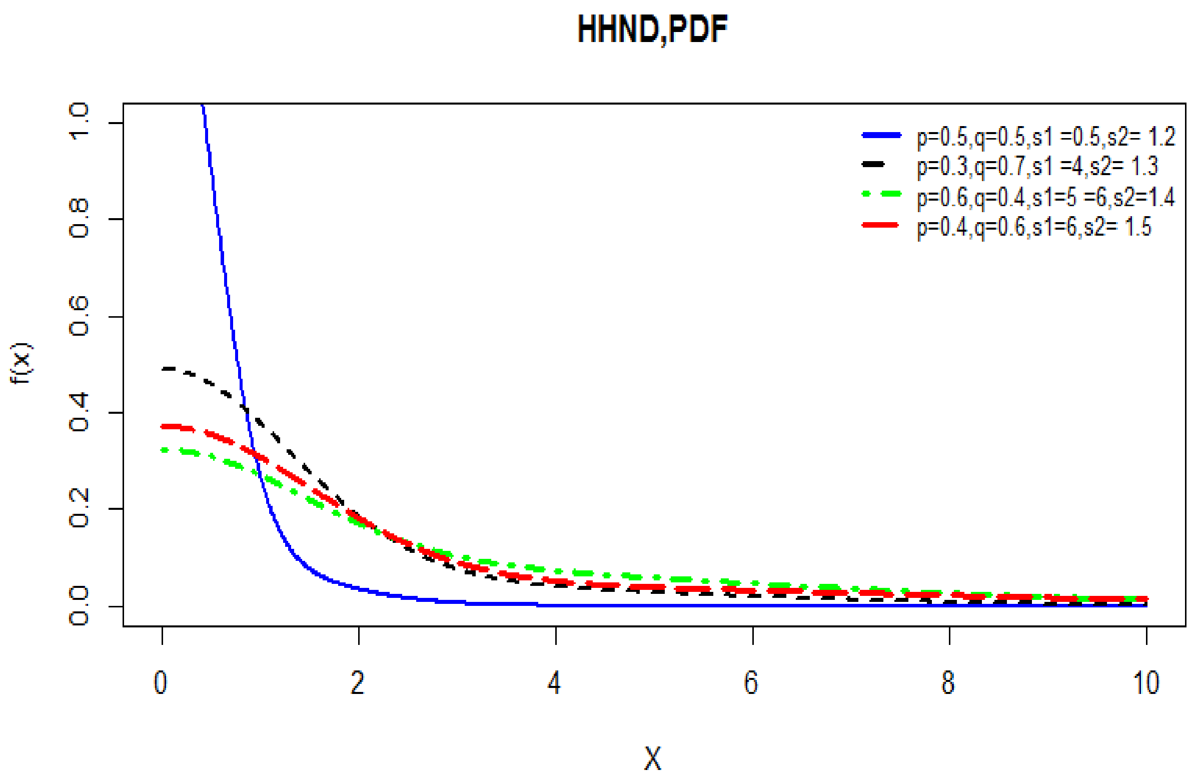
$$\mathbf{H(x)} = \frac{f_{HHND}(x;\sigma_1,\sigma_2)}{\mathbf{R(t)}} = \frac{p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}}{1-\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)}$$

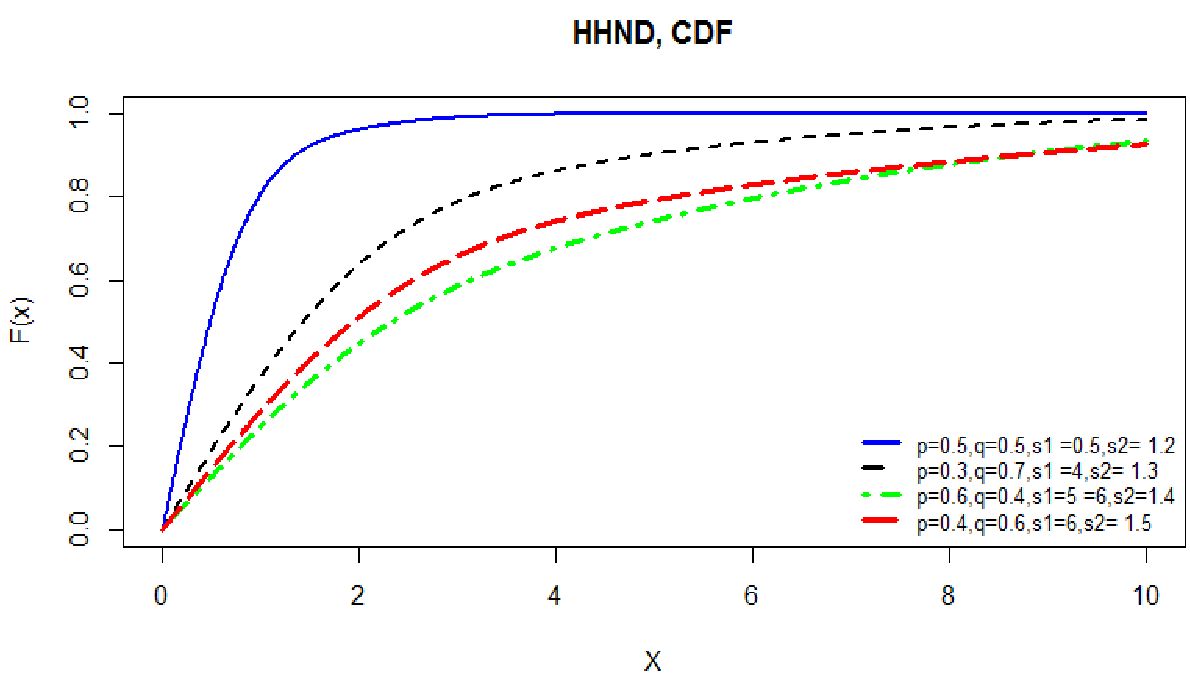
2.1 Method of Maximum Likelihood

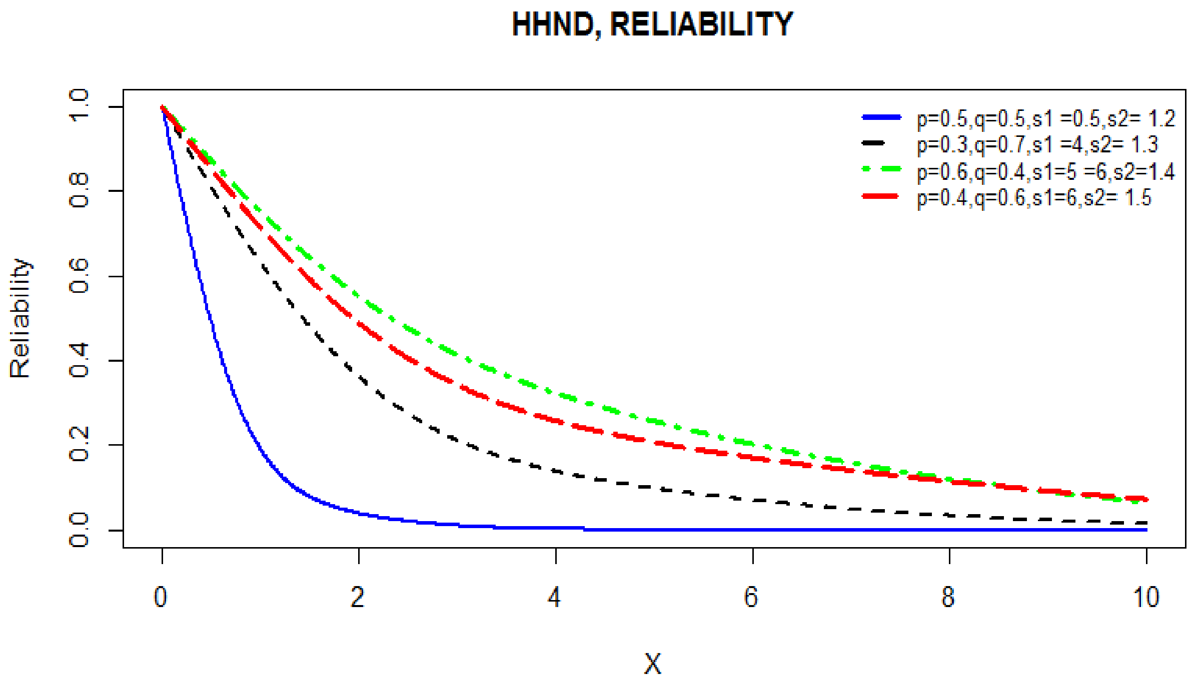
$$\mathbf{L}f_{HHND}(x;\ \sigma_1, \sigma_2) = \prod_{i=1}^{n} f_{HHND}(x;\ \sigma_1, \sigma_2)$$

$$\mathbf{L}f_{HHND}(x;\ \sigma_1, \sigma_2) = \prod_{i=1}^{n} p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}$$

$$lnLf_{HHND}(x;\ \sigma_1, \sigma_2) = \sum_{i=1}^{n} \ln\left[p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\right]$$


HHND,PDF
f(x)
X
p=0.5,q=0.5,s1 =0.5,s2= 1.2
p=0.3,q=0.7,s1 =4,s2= 1.3
p=0.6,q=0.4,s1=5 =6,s2=1.4
p=0.4,q=0.6,s1=6,s2= 1.5


HHND, CDF
F(x)
X
p=0.5,q=0.5,s1 =0.5,s2= 1.2
p=0.3,q=0.7,s1 =4,s2= 1.3
p=0.6,q=0.4,s1=5 =6,s2=1.4
p=0.4,q=0.6,s1=6,s2= 1.5



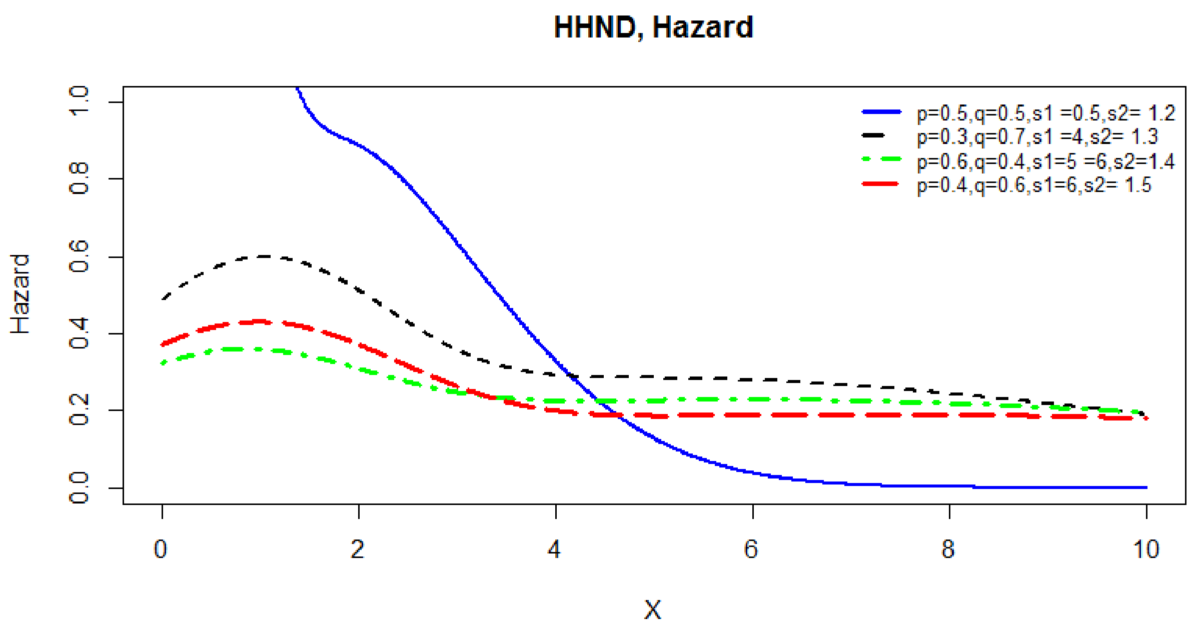


Fig1: PDF, CDF, Reliability and Hazard Function of Hyper halfnormal Distribution (HHND) at different parameters value

From the plots in **Fig1**, we can conclude that the HyperHalfNormal Distribution (HHND) is heavily tailed and skewed, the additional parameter increased the flexibility of the model exceeding beyond the scope of HalfNormal. It can be used to model non-normal data.

3.1 Derivation of Beta-HyperHalfNormal Distribution

From logit of beta by Jones[4], the mixture of Beta-hyper halfnormal distribution can be obtained. Let X be a random variable from the distribution with parameters as defined by Jones as shown by (16).

$$g(x) = \frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x) \qquad (16)$$

Where $g(x;\sigma_1,\sigma_2) = p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1{}^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2{}^2}}$ *and* $G(x;\sigma_1,\sigma_2) = perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)$ are the pdf and cdf of Hyper halfnormal distribution respectively. Then the Beta-hyper halfnormal Distribution is obtained by using (16) are shown by (17).

$$f_{BHHND}(x;a,b,\sigma_1,\sigma_2) = \frac{1}{B(a,b)}\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)^{a-1}\left(1-\left[perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right]\right)^{b-1} p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1{}^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2{}^2}},\ a,b,\sigma_1,\sigma_2 > 0;x>0 \qquad (17)$$

$$F_{BHHND}(x;a,b,\sigma_1,\sigma_2) = \int_0^x \frac{1}{B(a,b)}\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)^{a-1}\left(1-\left[perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right]\right)^{b-1} p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1{}^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2{}^2}} \qquad (18)$$

From (18), it follows that;

$F_{BHHND}(x;a,b,\sigma_1,\sigma_2)=\frac{1}{B(a,b)}\int_0^t P^{a-1}(1-P)^{b-1}\,dp$ where P is the cdf of parent distribution

By following the approach of incomplete beta function, which states B(x;α,β) = $\int_0^t x^{\alpha-1}(1-x)^{\beta-1}$. From this expression we can conclude that $\frac{1}{B(a,b)}\int_0^t P^{a-1}(1-P)^{b-1}\,dz = \frac{B(P;a,b)}{B(a,b)}$ = $I_P(a, b)$, where $I_P(a, b)$ is the regularized incomplete beta function and P is the cdf of the parent distribution.

$$F_{BHHND}(P) = I_P(a,b) = \frac{B(P;a,b)}{B(a,b)} \quad (19)$$

Since B(P; a, b) $= P^a\sum_{n=0}^{\infty}\frac{(1-b)}{n!(a+n)}P^n$, then (28) becomes (20)

$$F_{BHHND}(P) = \frac{B(P;a,b)}{B(a,b)} = \frac{P^a}{B(a,b)}\sum_{n=0}^{\infty}\frac{(1-b)}{n!(a+n)}P^n \quad (21)$$

$$F_{BHHND(P)}(x) = \frac{P^a}{B(a,b)}\left[\frac{1}{a}+\frac{1-b}{a+b}+------+\frac{(1-b)(2-b)(n-b)P^n}{n!(a+b)}\right] \quad (22)$$

Therefore (26) and (30) are the pdf and cdf of Beta-HyperHalfNormal Distribution respectively

3.1.2 Reliability

$$\mathbf{R(t) = 1} - F_{BHHND}(P) = 1-\frac{P^a}{B(a,b)}\sum_{n=0}^{\infty}\frac{(1-b)}{n!(a+n)}P^n$$

3.1.3 Hazard Function

$$\mathbf{H(x)}= \frac{f_{BHHND}(x;a,b,\sigma_1,\sigma_2)}{R(t)}$$

$$= \frac{\frac{1}{B(a,b)}\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right)+qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)^{a-1}\left(1-\left[perf\left(\frac{x}{\sigma_1\sqrt{2}}\right)+qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right]\right)^{b-1}p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}}+q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}}{\frac{B(a,b)-P^a\sum_{i=0}^{n}\frac{(1-b)P^n}{n!(a+n)}}{B(a,b)}}$$

$$= \frac{\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right)+qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)^{a-1}\left(1-\left[perf\left(\frac{x}{\sigma_1\sqrt{2}}\right)+qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right]\right)^{b-1}p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}}+q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}}{B(a,b)-P^a\sum_{i=0}^{n}\frac{(1-b)P^n}{n!(a+n)}}$$

3.2 Moment

According to Cordeiro and De castro[10], the moment generating function of γ generated beta distribution is display as (32)

$$M_x(t) = \frac{1}{B(a,b)} \sum_{i=1}^{n} (-1)^i \binom{b-1}{i} \rho(t, ai-1) \tag{23}$$

Where $\rho(t,r) = \int_{-\infty}^{\infty} e^{tx} [F(x)]^r f(x) dx$ (24)

$$M_x(t) = \frac{1}{B(a,b)} \sum_{i=1}^{n} (-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{tx} [F(x)]^r f(x) dx \tag{25}$$

$$M_x(t) = \frac{1}{B(a,b)} \sum_{i=1}^{n} (-1)^i \binom{b-1}{i} \int_{0}^{\infty} e^{tx} \left[perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right]^r p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_1{}^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}} e^{-\frac{x^2}{2\sigma_2{}^2}} dx$$

(26)

When $a = b = i = 1$, the mgf of BHHND becomes the mgf of the parent distribution (HyperHalfNormal).

3.3 Maximum Likelihood Method

From the method developed by Cordeiro & De Castro[10] for beta-generated family; the loglikelihood estimation method for parameter, $\beta = (a,b,c,\tau)$, estimation, where τ is the parameter vector of the baseline distribution.

$$l(\theta) = nlogc - nlog[B(a,b)] + \sum_{i=1}^{n} logf(x_i,\tau) + (a-1)\sum_{i=1}^{n} logF(x_i,\tau) + (b-1)\sum_{i=1}^{n} \log(1-F(x_i,\tau))$$

(27)

The generalized distribution reduces to the class of beta generated distribution when c=1, and the parameter vector β now $\beta = (a,b,\sigma_1,\sigma_2)$.

$$l(\beta) = -n\log[B(a,b)] + \sum_{i=1}^{n} \log f(x_i,\tau) + (a-1)\sum_{i=1}^{n} \log F(x_i,\tau) + (b-1)\sum_{i=1}^{n} \log\left(1 - F(x_i,\tau)\right) \tag{28}$$

By making necessary substitution, (28) becomes (29).

$$l(\beta) = -n\log[B(a,b)] + \sum_{i=1}^{n} \log\left(p\frac{\sqrt{2}}{\sigma_1\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_1^2}} + q\frac{\sqrt{2}}{\sigma_2\sqrt{\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\right) + (a-1)\sum_{i=1}^{n} \log\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right) + (b-1)\sum_{i=1}^{n} \log\left(1 - \left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)\right) \tag{29}$$

$$\frac{\partial L(\beta)}{\partial a} = \frac{-n\acute{\Gamma}(a)}{\Gamma(a)} + \frac{\acute{\Gamma}(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n} \log\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right) \tag{30}$$

$$\frac{\partial L(\beta)}{\partial b} = \frac{-n\acute{\Gamma}(b)}{\Gamma(a)} + \frac{\acute{\Gamma}(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n} \log\left(1 - \left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)\right) \tag{31}$$

=

$$\frac{\partial L(\beta)}{\partial \sigma_1} = \sum_{i=1}^{n}\left(\frac{-\sigma_2 p e^{\frac{x^2}{2\sigma_2^2}}(\sigma_1^2 - x^2)}{\sigma_1^3\left(\sigma_1 q e^{\frac{x^2}{2\sigma_1^2}} + \sigma_2^2 p e^{\frac{x^2}{2\sigma_2^2}}\right)}\right) + \sum_{i=1}^{n}\left(\frac{-\sqrt{\frac{2}{\pi}}\,xpe^{-\frac{x^2}{\sigma_1^2}}}{\sigma_1^2\left(perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)}\right) + \sum_{i=1}^{n}\left(\frac{-\sqrt{\frac{2}{\pi}}\,xpe^{-\frac{x^2}{\sigma_1^2}}}{\sigma_1^2\left(-perf\left(\frac{x}{\sigma_1\sqrt{2}}\right) - qerf\left(\frac{x}{\sigma_2\sqrt{2}}\right) + 1\right)}\right)$$

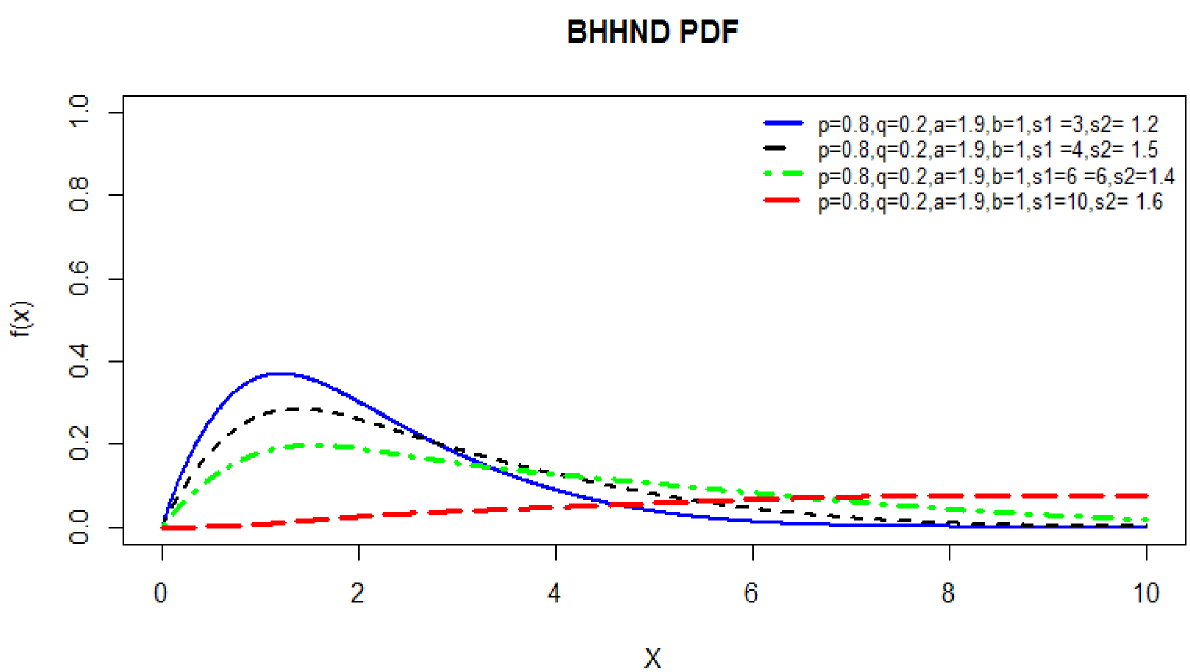
=

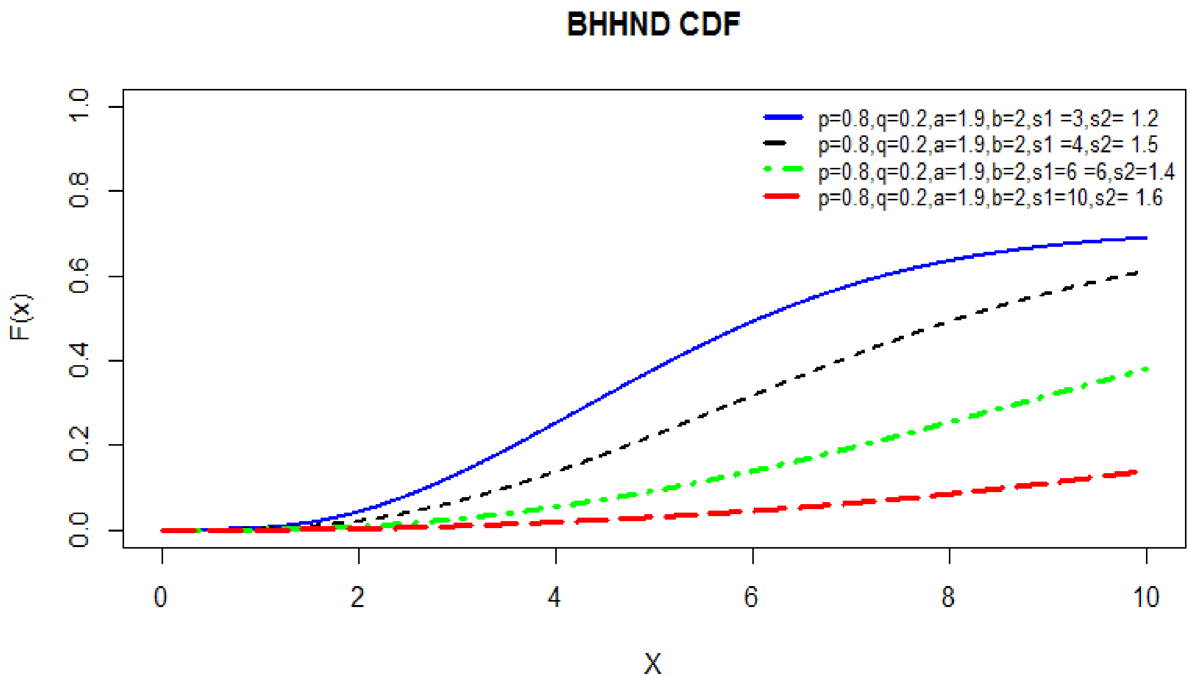
$$\frac{\partial L(\beta)}{\partial \sigma_2} = \sum_{i=1}^{n}\left(\frac{-\sigma_1 q e^{\frac{x^2}{2\sigma_1^2}}(\sigma_2^2 - x^2)}{\sigma_2^3\left(\sigma_2 q e^{\frac{x^2}{2\sigma_2^2}} + \sigma_1^2 p e^{\frac{x^2}{2\sigma_1^2}}\right)}\right) + \sum_{i=1}^{n}\left(\frac{\sqrt{\frac{2}{\pi}}\, x q e^{-\frac{x^2}{\sigma_2^2}}}{\sigma_2^2\left(p\,erf\left(\frac{x}{\sigma_1\sqrt{2}}\right) + q\,erf\left(\frac{x}{\sigma_2\sqrt{2}}\right)\right)}\right) + \sum_{i=1}^{n}\left(\frac{-\sqrt{\frac{2}{\pi}}\, x q e^{-\frac{x^2}{\sigma_2^2}}}{\sigma_2^2\left(-p\,erf\left(\frac{x}{\sigma_1\sqrt{2}}\right) - q\,erf\left(\frac{x}{\sigma_2\sqrt{2}}\right) + 1\right)}\right)$$

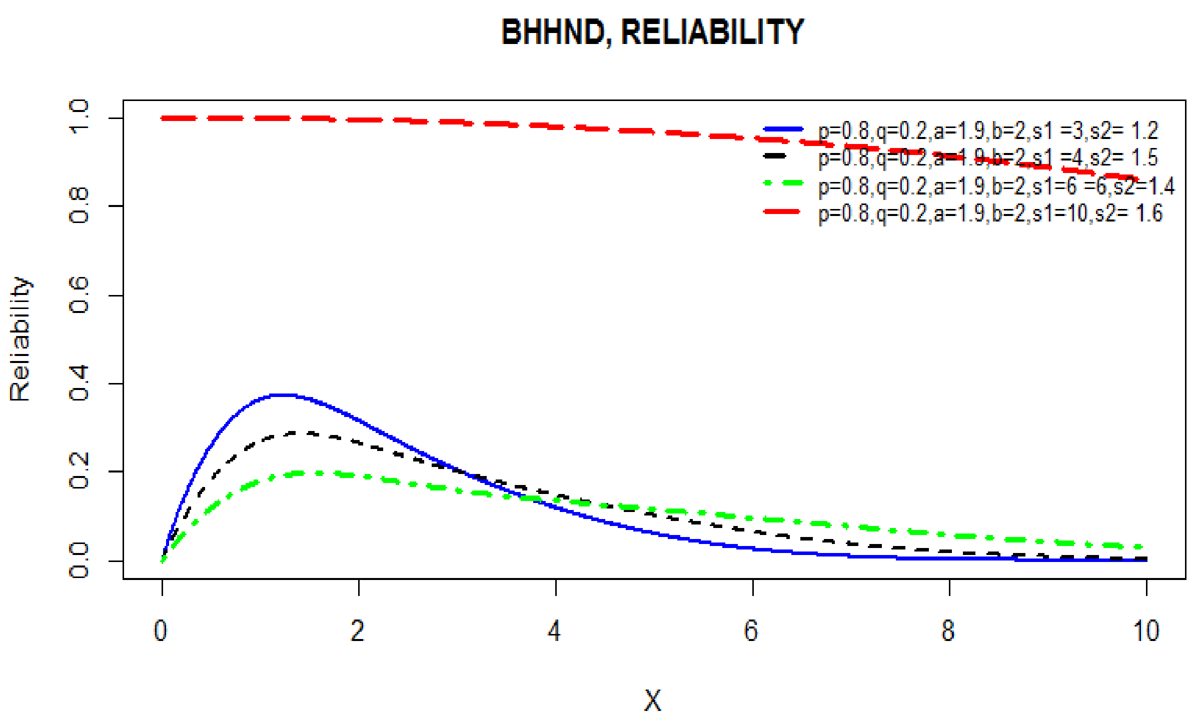
(32)

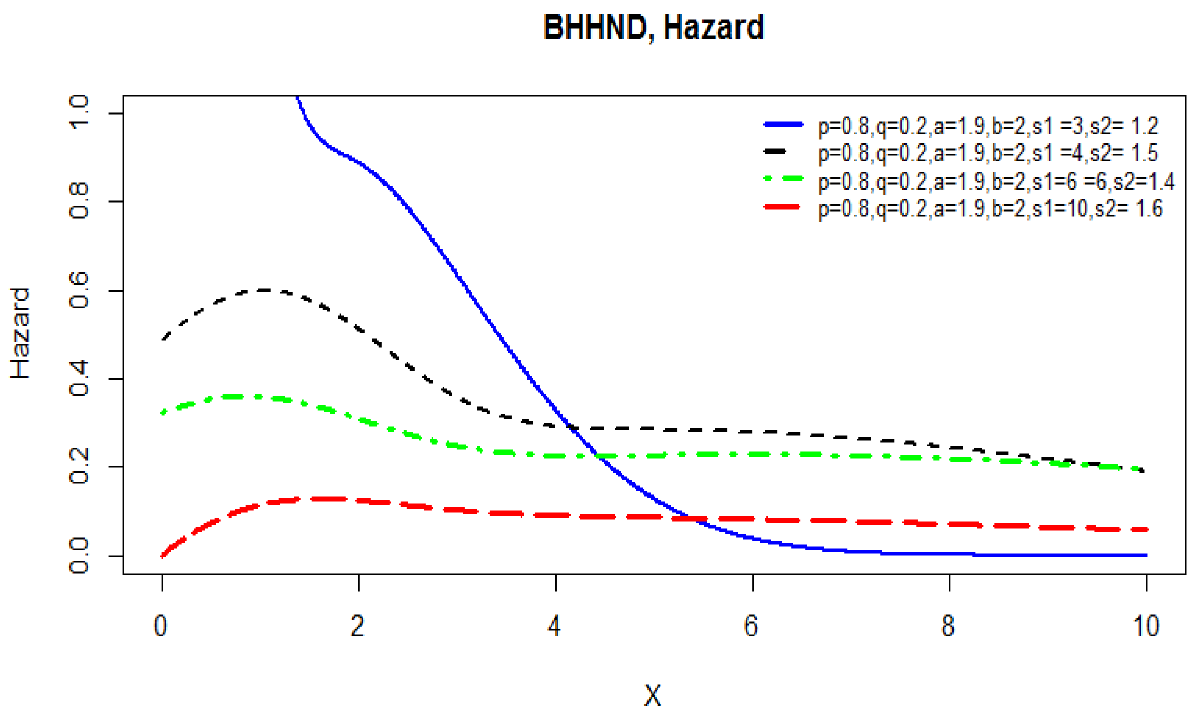
The parameter $\hat{a}, \hat{b}, \widehat{\sigma_1}$ *and* $\widehat{\sigma_2}$ can be obtained using the numerical method (Newton Raphson Method) and fisher information matrix can be used to test hypothesis about the significant of the additional parameters.

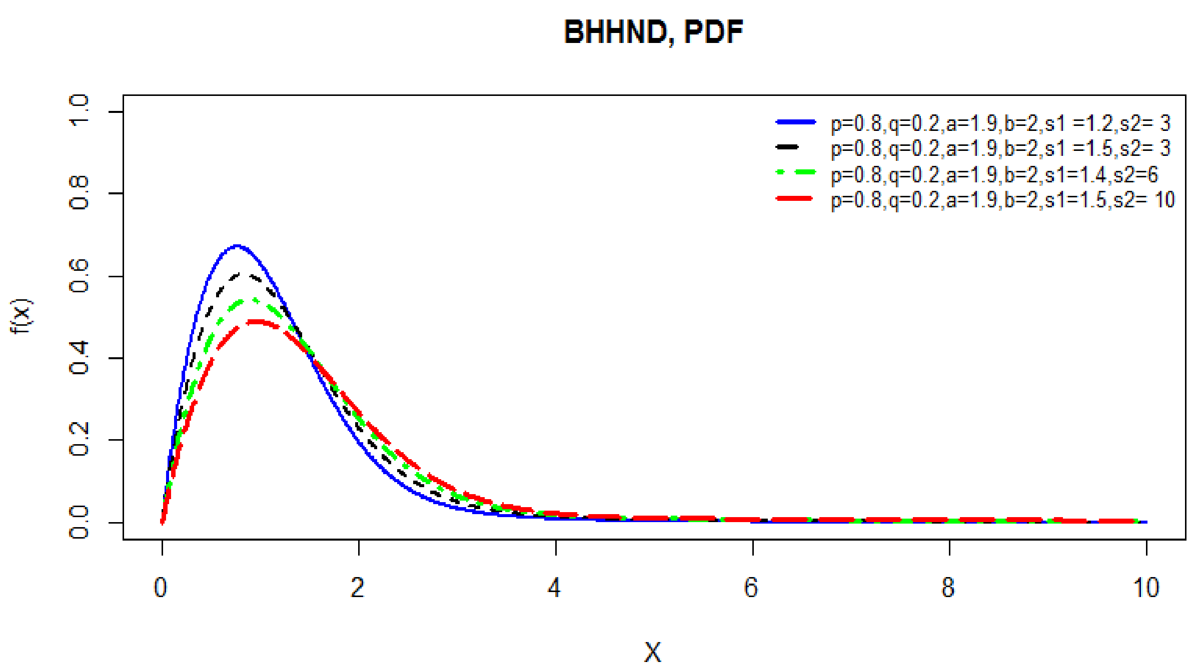
3.4 Plot of Beta-Hyper-halfnormal Distribution (BHHND)

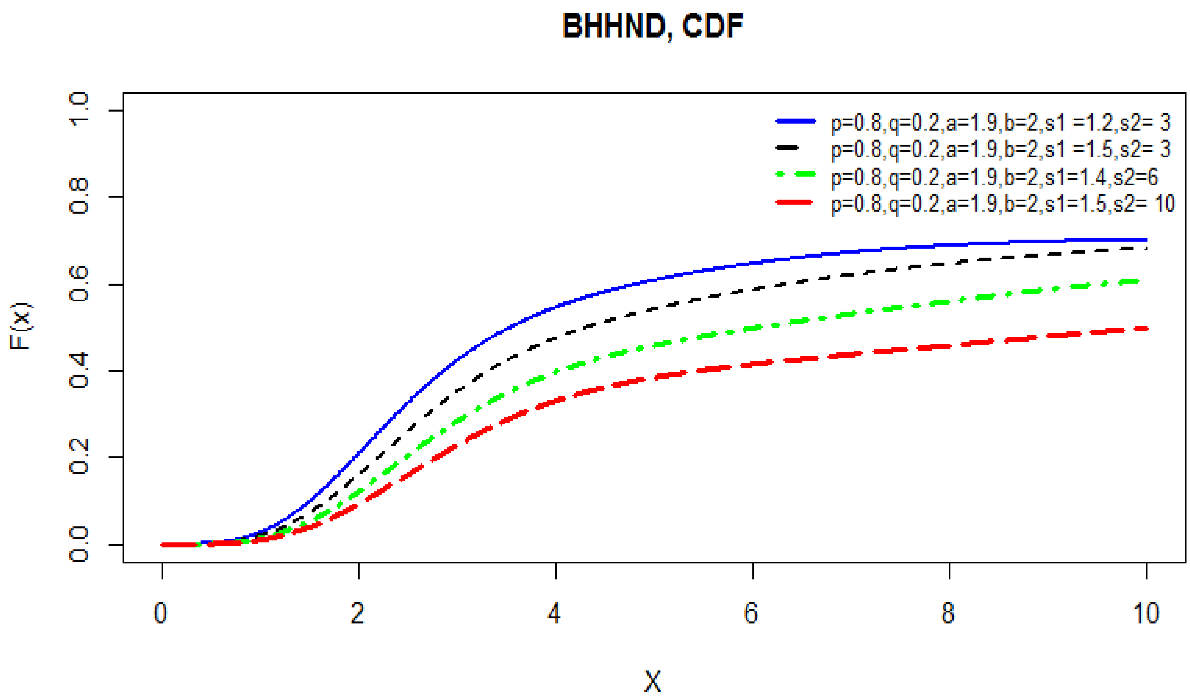


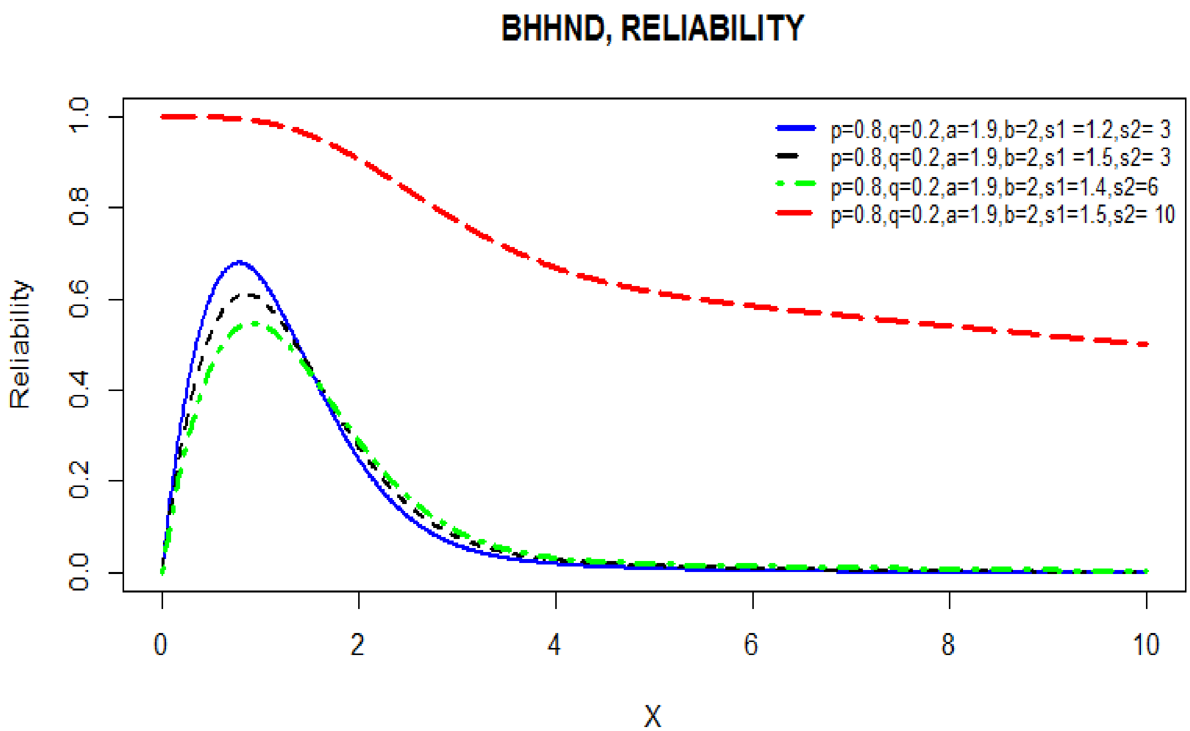

BHHND CDF
F(x)
X
p=0.8,q=0.2,a=1.9,b=2,s1 =3,s2= 1.2
p=0.8,q=0.2,a=1.9,b=2,s1 =4,s2= 1.5
p=0.8,q=0.2,a=1.9,b=2,s1=6 =6,s2=1.4
p=0.8,q=0.2,a=1.9,b=2,s1=10,s2= 1.6

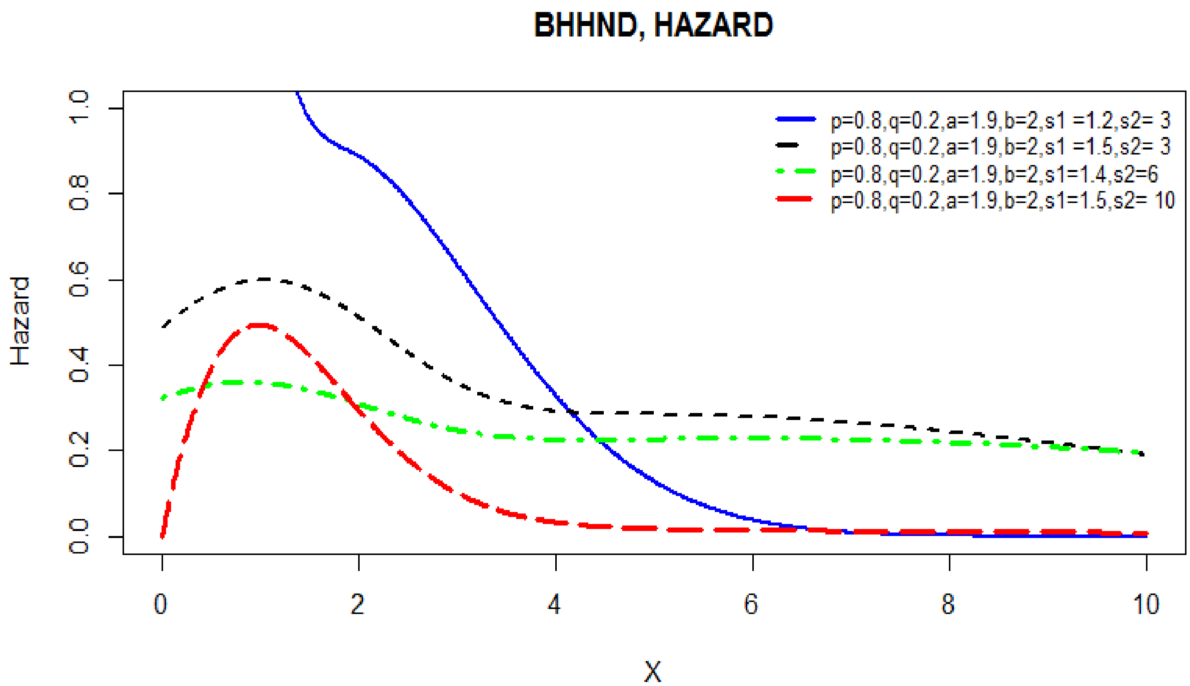

BHHND, RELIABILITY
Reliability
X
p=0.8,q=0.2,a=1.9,b=2,s1 =3,s2= 1.2
p=0.8,q=0.2,a=1.9,b=2,s1 =4,s2= 1.5
p=0.8,q=0.2,a=1.9,b=2,s1=6 =6,s2=1.4
p=0.8,q=0.2,a=1.9,b=2,s1=10,s2= 1.6

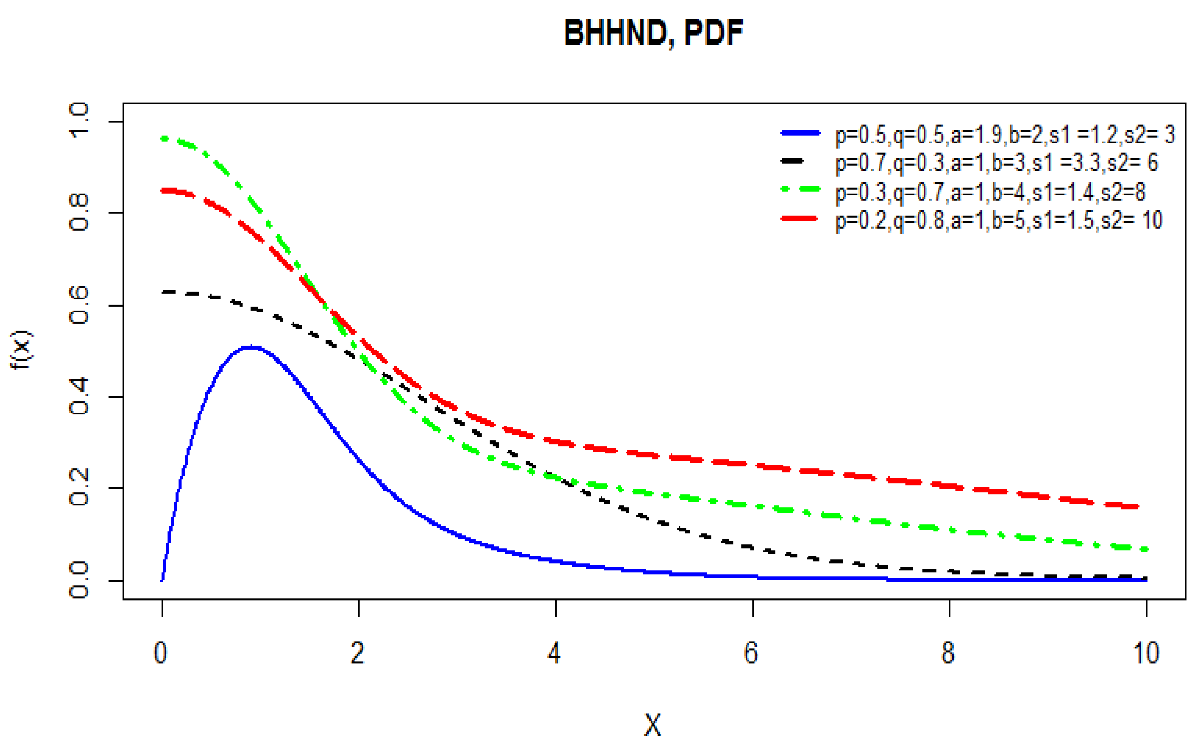

BHHND, Hazard
Hazard
X
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
p=0.8,q=0.2,a=1.9,b=2,s1 =3,s2= 1.2
p=0.8,q=0.2,a=1.9,b=2,s1 =4,s2= 1.5
p=0.8,q=0.2,a=1.9,b=2,s1=6 =6,s2=1.4
p=0.8,q=0.2,a=1.9,b=2,s1=10,s2= 1.6

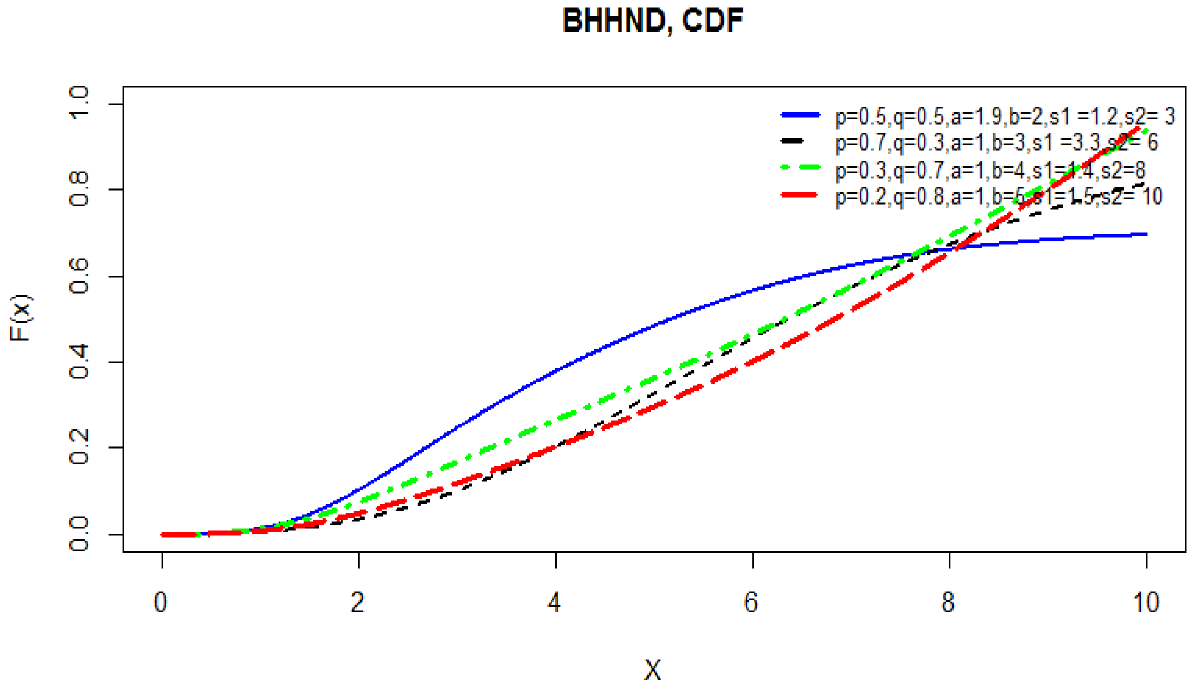

BHHND, PDF
f(x)
X
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
p=0.8,q=0.2,a=1.9,b=2,s1 =1.2,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1 =1.5,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1=1.4,s2=6
p=0.8,q=0.2,a=1.9,b=2,s1=1.5,s2= 10

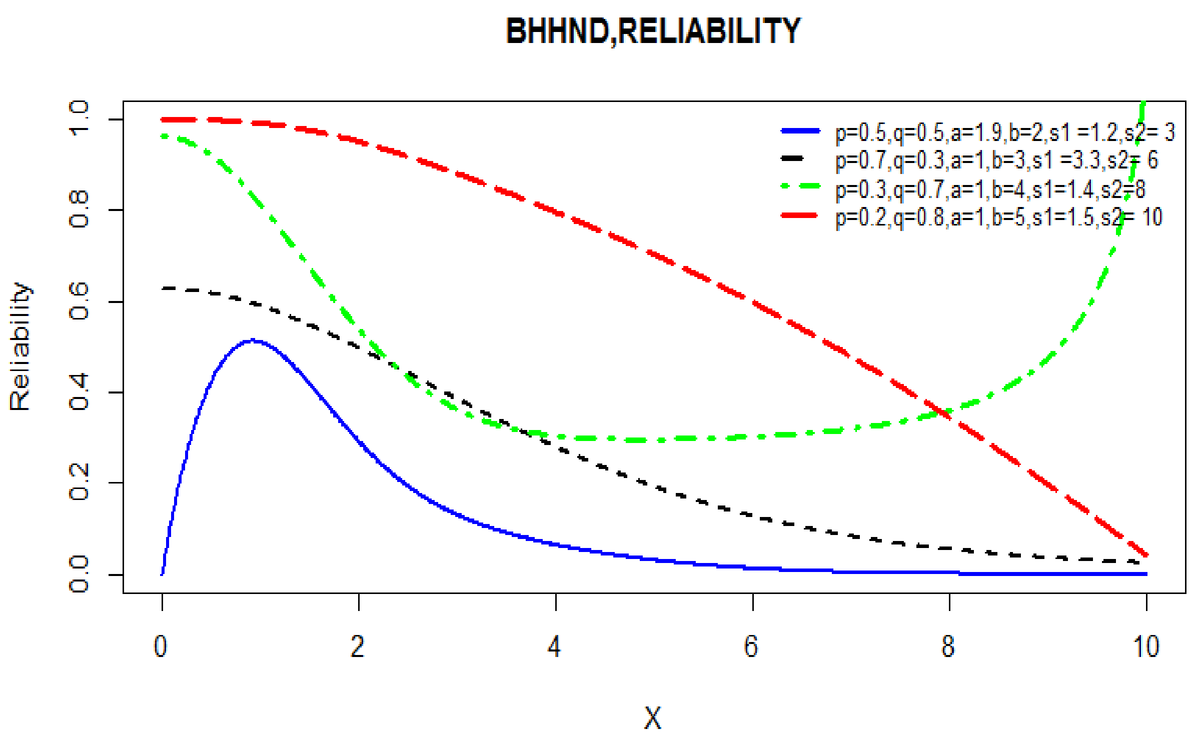

BHHND, CDF
F(x)
X
p=0.8,q=0.2,a=1.9,b=2,s1 =1.2,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1 =1.5,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1=1.4,s2=6
p=0.8,q=0.2,a=1.9,b=2,s1=1.5,s2= 10


BHHND, RELIABILITY
Reliability
X
p=0.8,q=0.2,a=1.9,b=2,s1 =1.2,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1 =1.5,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1=1.4,s2=6
p=0.8,q=0.2,a=1.9,b=2,s1=1.5,s2= 10


BHHND, HAZARD
p=0.8,q=0.2,a=1.9,b=2,s1 =1.2,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1 =1.5,s2= 3
p=0.8,q=0.2,a=1.9,b=2,s1=1.4,s2=6
p=0.8,q=0.2,a=1.9,b=2,s1=1.5,s2= 10
Hazard
X


BHHND, PDF
p=0.5,q=0.5,a=1.9,b=2,s1 =1.2,s2= 3
p=0.7,q=0.3,a=1,b=3,s1 =3.3,s2= 6
p=0.3,q=0.7,a=1,b=4,s1=1.4,s2=8
p=0.2,q=0.8,a=1,b=5,s1=1.5,s2= 10
f(x)
X


BHHND, CDF
p=0.5,q=0.5,a=1.9,b=2,s1 =1.2,s2= 3
p=0.7,q=0.3,a=1,b=3,s1 =3.3,s2= 6
p=0.3,q=0.7,a=1,b=4,s1=1.4,s2=8
p=0.2,q=0.8,a=1,b=5,s1=1.5,s2= 10
F(x)
X


BHHND,RELIABILITY
p=0.5,q=0.5,a=1.9,b=2,s1 =1.2,s2= 3
p=0.7,q=0.3,a=1,b=3,s1 =3.3,s2= 6
p=0.3,q=0.7,a=1,b=4,s1=1.4,s2=8
p=0.2,q=0.8,a=1,b=5,s1=1.5,s2= 10
Reliability
X

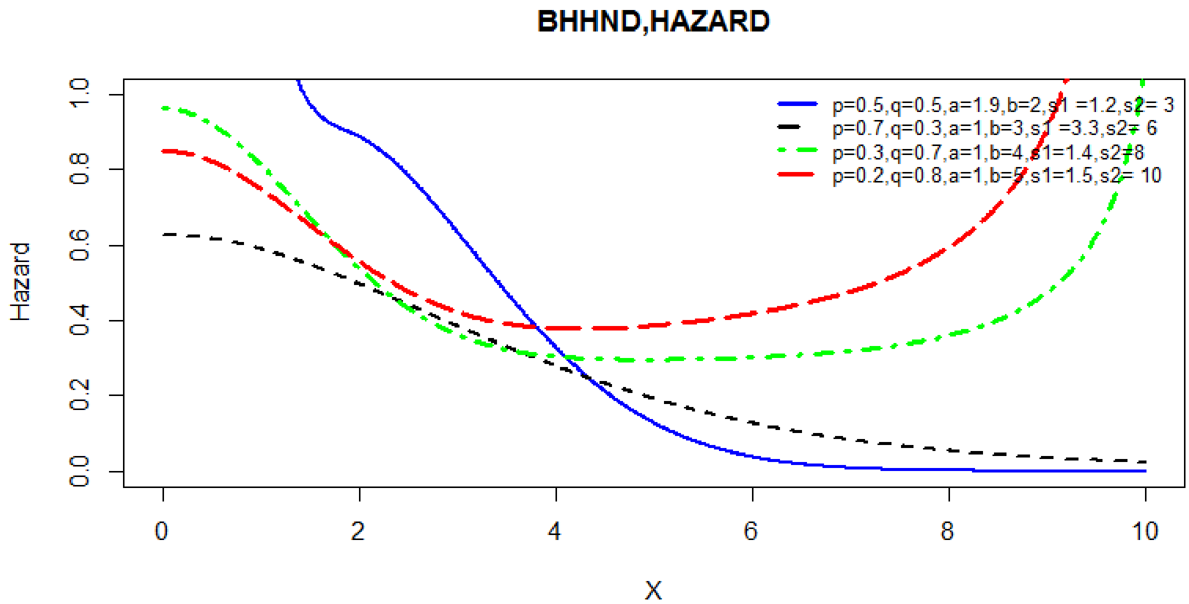


Fig2 : PDF, CDF, Reliability and Hazard Function of Beta-HyperHalfNormal Distribution (BHHND) at different parameters value

From the plots in **Fig 2**, the plot of PDF shows that the BHHND is heavily tailed, skewed and has mode. Its hazard plot showed that the BHHND has bathtube hazard. It can therefore be used to model processes with bathtube hazard function, skewed and or heavily tailed.

4.1 Application of Hyper-halfnormal Distribution (HHND) and Beta-hyper halfnormal Distribution (BHHND)

Consider the data set presented by Aarset[9]. The data describe lifetimes of 50 industrial devices put on life test at time zero. The data from Aarset[9] is shown below. This data is used in this research and BHHND, HHND and HND are fitted to the data.

0.1,0.2,1,1,1,1,1,2,3,6,7,11,12,18,18,18,18,18,21,32,36,40,45,46,47,50,55,60,63,63,67,67,67,67
72,75,79,82,82,83,84,84,84,85,85,85,85,85,86,86

4.1.1 Accessing Distributionof the Data

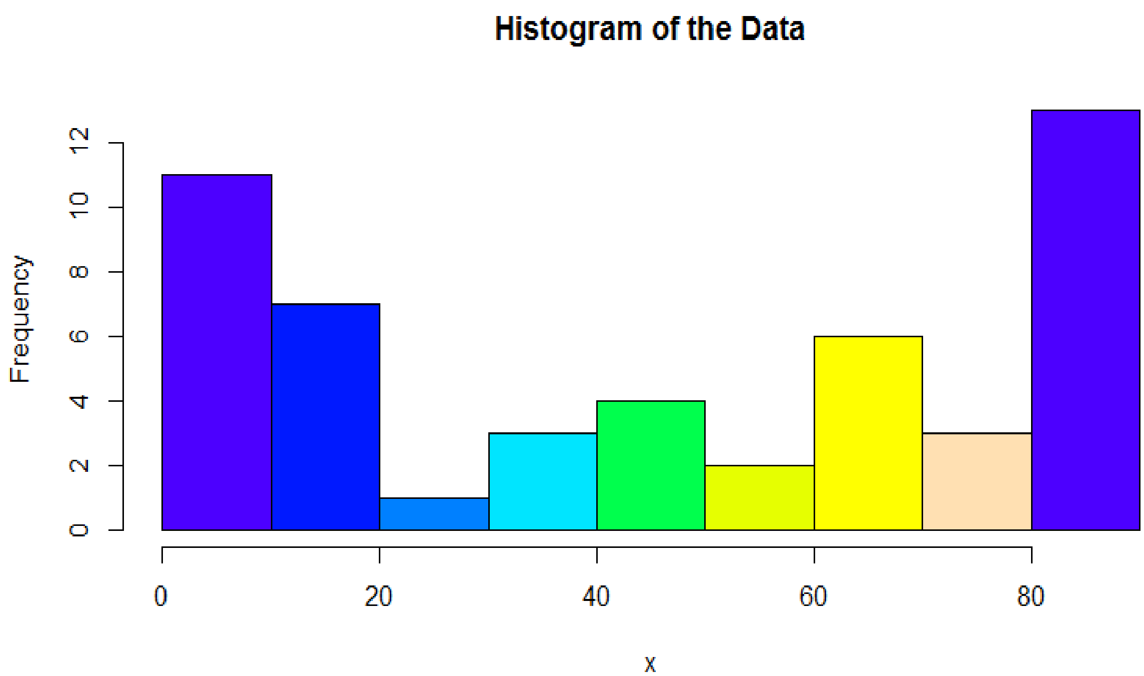


Fig 3: Accessing the distribution of the lifetime data

From **Fig 3** above, we can say that the data is non-normal and the distribution of the data is approximately U-shape, therefore the process that generated the data can be modeled using hybrid Beta-hyper-halfnormal Distribution (BHHND). Since the model is said to have bathtube hazard, then BHHND is a suitable model for the data.

Table 1: Fitting Halfnormal Distribution (HND), HHND and BHHND to the lifetime Data

Parameter p = 0.4,q =0.6	**HHND**	**BHHND**	**HND**
σ_1	0.02059014(0.00224717)	0.1187801(0.031952)	56.070(4.194)
σ_2	1.07785600(0.39998258)	6.00(0.2298813)	
a		4.555746e-03 (1.109552e-05)	
b		2.2011 (8.390270e-03)	
	W = 12.06639	**W** =111.0877	
Coefficient of Variation	**CV** = 0.8375417		**CV** = 0.7559289

From **Table 1** above, the parameter of the model is estimated and the coefficient of variation of HHND is greater that the coefficient of variation of HND. The distribution (HHND) is the hyper version of Halfnormal Distribution because its coefficient of variation is larger than that of HalfNormal. The definition of Hyper halfnormal depends on the prevailing value of p and q (mixing proportion). For q > p, the distribution is Hyper halfnormal and it is Hypo-halfnormal if otherwise.

4.2 WALD TEST: Test for Significance of $\hat{a}$ and $\tilde{b}$ in the Model

Hypothesis

H_0: $a = b = 0 \ , \alpha = 0.05$

H_1: $a \neq b \neq 0$

Decision Rule:

Accept H_0 if $w < {\chi_q}^2$ *, otherwise do not accept* q is the number of parameters in the model or the number of rows of the variance-covariance matrix

$\chi_q{}^2$, at q = 4, is 9.49

Decision

Since the w (111.0877) > $\chi_q{}^2$, then we have statistical reason not to accept H_0 and conclude that the shape parameter $\hat{a}$ and $\hat{b}$ contributed significantly to the modeling behavior or efficiency of the hybrid BHHND model. The parameters have increased the flexibility of BHHND over HHND in modeling data with bathtub distribution.

4.2.2 Test for Significance of $\widehat{\sigma_2}$ in the Model (HHND)

Hypothesis

H_0: $\widehat{\sigma_2} = 0$, $\alpha = 0.05$

H_1: $\widehat{\sigma_2} \neq 0$

$\chi_q{}^2$, at q = 2, is 5.99

Decision

Since the w (12.06639) > $\chi_q{}^2$, then we have statistical reason not to accept H_0 and conclude that the shape parameter $\widehat{\sigma_2}$ has also contributed significantly to the modeling efficiency of the hybrid HHND model. The parameters have increased the flexibility of HHND over HND in modeling data with heavily tailed distribution (Non-normal data).

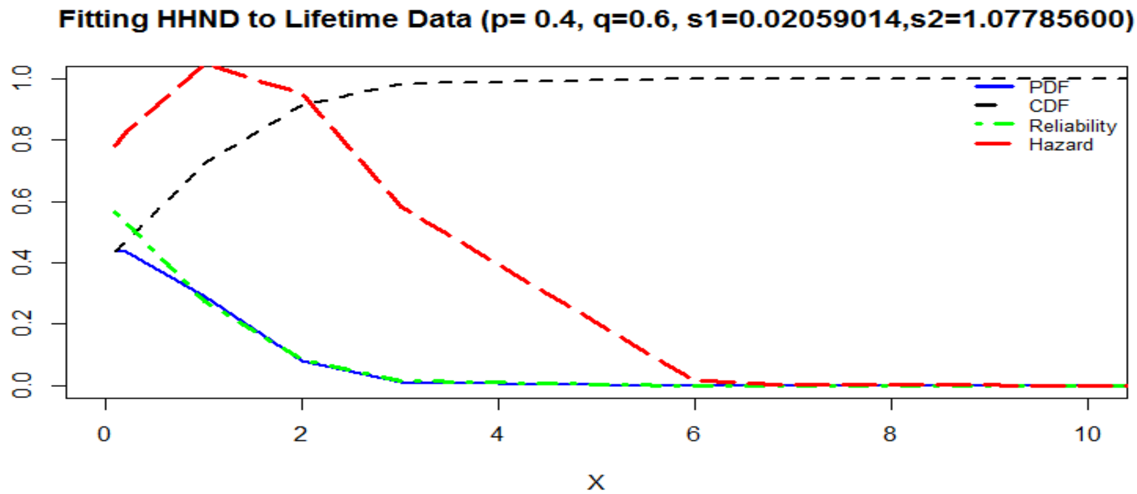
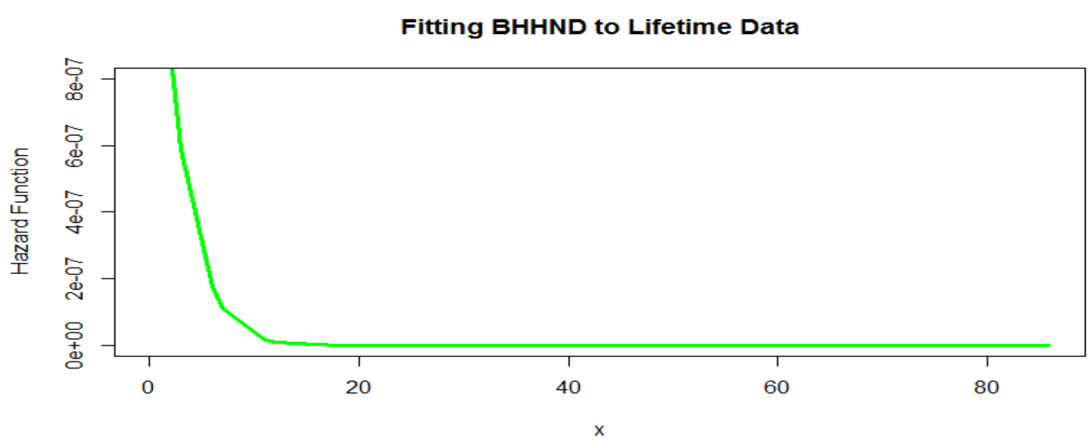


Fig 4: Plot of HHND at the estimated parameter values.

From Fig. 4 above, we can deduce that the hazard curve is not bathtub, thus making hybrid HHND unsuitable model as compared to distribution with bathtub hazard (BHHND) when fitted to data with bathtub distribution (Lifetime Data)



From Fig. 5 above, we can conclude that the BHHND fit the lifetime data the most as compared to hybrid HHND. This is because the hazard plot models the distribution of the data.

4.3 Conclusions

It has been established that BHHND provides a better fit to lifetime data with bathtube hazard as compared to HNND and HND and for $q > p$, we have hyper-halfnormal distribution which has its coefficient of variation greater than that of HND. HHND is found to be unimodal, heavily tailed with bathtube hazard and this has increased its flexibility in capturing lifetime data reasonably well.

References

[1] T.K. Pogany and S. Nadarajah, On the Convolution of normal and t random variables, Statistics: A Journal of Theoretical and Applied Statistics, 47(2012) 1363 -1369

[2] P.V. Orlik and S.S. Rapport, Traffic Performance and mobility modeling of cellular communication with mixed platform and highly variable mobilities, Proceedings of the IEEE, 86(1998) 1464-1479.

[3] A.A Akomolafe and A. Maradesa, Beta-halfnormal distribution and its properties, international Journal of Advance Research and Publication, 1(2017) 17-22.

[4] M.C. Jones, Families of distributions arising from distribution of order statistics, Test, 13(2004) 1-43.

[5] A. Akinsete, F. Famoye and C. Lee, The beta-pareto distribution, Journal of Theoretical and Applied Statistics, 42(2008) 547-563

[6] N. Eugene, C. Lee and F. Famoye, Beta-normal Distribution and its applications, Communications in Statistics-Theory and Methods, 31(2002) 497-512.

[7] N.B. Idowu, Ikeggwu and M. Emmanuel, Beta-weighted weibull Distribution:some properties and application to bladder cancer data, J. Appl. Computat. Math., 2(2013) 1-6.

[8] M.R. Miroslav and S. Nadarajah, A new lifetime distribution, Journal of Statistical Computation and Simulation 84(2014) 135- 150.

[9] V.M Aarset, How to identify bathtub hazard rate, IEEE Transactions on Reliability, 36(1987) 106-108.

[10] G.M. Cordeiro and M. Decastro, A new family of generalized distributions, Journal of Statistical Computation and Simulation, 81(2011) 883-898.