

Applications of the Operator T in q -Polynomials

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<u>ARTICLE INFO</u>	ABSTRACT
<p>Keywords Cauchy polynomials, bivariate Rogers- Szegő polynomials, the generating function, Rogers formula, Mehler's formula</p>	<p>In this paper, we define the polynomials $V_n(a, b, c, f, x, y)$. In order to determine the generating function, Rogers' formula, Mehler's formula, and their extensions for polynomials $V_n(a, b, c, f, x, y)$, we utilize the q-exponential operator T. Some results for the Cauchy polynomials $P_n(x, y)$ and the bivariate Rogers-Szegő polynomials $h_n(x, y q)$ are obtained by inserting special values into the identities of the polynomials $V_n(a, b, c, f, x, y)$.</p>

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1. Introduction

In this paper we will use the standard notations for basic hypergeometric series given in [1], we assume that $|q| < 1$.

Let a be a complex variable. The q -shifted factorial is defined by [1], [2]

$$(a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & \text{if } n = 1, 2, \dots \end{cases}$$

We define

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The following notation is used for the multiple q -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n, \quad n = 0, 1, 2, \dots$$

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_m; q)_\infty.$$

The generalized basic hypergeometric series is defined by [1].

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r \in \mathbb{C}$; $b_1, \dots, b_s \in \mathbb{C} \setminus \{q^{-k}, k \in \mathbb{N}\}$ are assumed to be such that none of the denominator factors evaluate to zero. This series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$.

The case $r = s + 1$ is the most important class of series

$${}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{s+1}; q)_n}{(q, b_1, \dots, b_s; q)_n} x^n, \quad |x| < 1.$$

For $n, k \in \mathbb{N}$, the q -binomial coefficient is defined by [1]



$$[n]_k = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

One of the most important identities is the Cauchy identity [1]

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1. \quad (1)$$

Euler found the following special case of Cauchy identity [1]:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_{\infty}}, \quad |x| < 1. \quad (2)$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = (-x; q)_{\infty}. \quad (3)$$

Jackson's transformation formula is [1]

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} {}_2\phi_2 \left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz \right), \quad |z| < 1. \quad (4)$$

The Cauchy polynomials are defined by [3], [4]:

$$P_n(x, y) = \left(\frac{y}{x}; q\right)_n x^n = (x - y)(x - qy) \cdots (x - q^{n-1}y), \quad (5)$$

which have the generating function [5,6,7]:

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}, \quad |xt| < 1. \quad (6)$$

Mehler’s formula for $P_n(x, y)$ is [6], [8]:

$$\sum_{n=0}^{\infty} P_n(x, y)P_n(z, w) \frac{t^n}{(q; q)_n} = \frac{(xwt; q)_{\infty}}{(xzt; q)_{\infty}} {}_1\phi_1\left(\begin{matrix} \frac{w}{z} \\ xwt \end{matrix}; q, yzt\right), \quad |xzt| < 1. \quad (7)$$

Rogers formula for $P_n(x, y)$ is [9]

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(yt; q)_{\infty}}{(xs, xt; q)_{\infty}} {}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix}; q, ys\right), \quad (8)$$

where $\max\{|xs|, |xt|\} < 1$.

The bivariate Rogers-Szegő polynomials $h_n(x, y|q)$ is defined as :

$$h_n(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y), \quad (9)$$

which have the generating function [4]

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}}, \quad \max\{|t|, |xt|\} < 1. \quad (10)$$

Mehler’s Formula for $h_n(x, y|q)$ is [4]

$$\sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q) \frac{t^n}{(q; q)_n} = \frac{(yt, vxt; q)_{\infty}}{(t, xt, uxt; q)_{\infty}} {}_3\phi_2\left(\begin{matrix} y, xt, \frac{v}{u} \\ yt, vxt \end{matrix}; q, ut\right), \quad (11)$$

provided that $\max\{|t|, |xt|, |ut|, |uxt|\} < 1$.

Roger’s Formula for $h_n(x, y|q)$ is [4]

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(ys; q)_{\infty}}{(s, xs, xt; q)_{\infty}} {}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t\right), \quad (12)$$

provided that $\max\{|s|, |xs|, |xt|, |t|\} < 1$.

The q - differential operator D_q is defined by [10,11,12]

$$D_q f(x) = \frac{f(x) - f(xq)}{x}.$$

The Leibniz rule for D_q is [10]:

$$D_q^n \{f(x)g(x)\} = \sum_{k=0}^n \binom{n}{k} q^{k(k-n)} D_q^k \{f(x)\} D_q^{n-k} \{g(xq^k)\}. \tag{13}$$

The following results are easy to verify:

$$D_q^k \{P_n(x, y)\} = \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y). \tag{14}$$

$$D_q^k \left\{ \frac{1}{(xt; q)_\infty} \right\} = \frac{t^k}{(xt; q)_\infty}. \tag{15}$$

In 2016, inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [13] introduced the generalized q -exponential operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$ as follows:

$$\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) = \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} (fD_q)^k. \tag{16}$$

The following is how this paper is organized: In Section 2, we'll look at some identities for the

q -exponential operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$. Section 3 presents the generating function and its

extension for the polynomials $V_n(a, b, c, f, x, y)$ using the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$. Section 4

establishes Mehler's formula and its extension for the polynomials $V_n(a, b, c, f, x, y)$. The operator method to Rogers' formula for polynomials $V_n(a, b, c, f, x, y)$ will be used in Section 5.

Identities for the q -Exponential Operator \mathbb{T} .2

In this section, we provide some identities for the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$.

Theorem 2.1 Let the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$ be defined as in (16), then we have

$$\begin{aligned} & \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{1}{(xs, xt; q)_\infty} \right\} \\ &= \frac{1}{(xs, xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(a, b; q)_{i+k} (xs; q)_i}{(c; q)_{i+k} (q; q)_i} (ft)^i, \quad \max\{|xs|, |xt|\} < 1. \end{aligned} \tag{17}$$

Proof.

$$\begin{aligned} & \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{1}{(xt, xs; q)_\infty} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} f^k D_q^k \left\{ \frac{1}{(xt, xs; q)_\infty} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} f^k \sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix} q^{i(i-k)} D_q^i \left\{ \frac{1}{(xt; q)_\infty} \right\} D_q^{k-i} \left\{ \frac{1}{(xsq^i; q)_\infty} \right\} \quad (\text{by using (13)}) \\ &= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} f^k \sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix} q^{i(i-k)} \frac{t^i (q^i s)^{k-i}}{(xt; q)_\infty (xsq^i; q)_\infty} \\ &= \square \frac{1}{(xt, xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(a, b; q)_{i+k} (xs; q)_i}{(c; q)_{i+k} (q; q)_i} (ft)^i. \end{aligned}$$

• Setting $s = 0$ in equation (17), we get

$$\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{1}{(xt; q)_\infty} \right\} = \frac{1}{(xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, ft \right), \quad |xt| < 1. \tag{18}$$

Theorem 2.2 Let the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$ be defined as in (16), then

$$\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ P_n(x, y) \frac{(ux; q)_\infty}{(sx; q)_\infty} \right\} = \frac{(ux; q)_\infty}{(sx; q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j q^{k(k-j)} \begin{bmatrix} j \\ k \end{bmatrix} \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y) s^{j-k} \frac{(u/s; q)_{j-k} (sx; q)_k}{(ux; q)_j} \quad |sx| < 1. \tag{19}$$

Proof.

$$\begin{aligned} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ P_n(x, y) \frac{(ux; q)_{\infty}}{(sx; q)_{\infty}} \right\} &= \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j D^j \left\{ P_n(x, y) \frac{(ux; q)_{\infty}}{(sx; q)_{\infty}} \right\} \\ &= \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} D_q^j \{ P_n(x, y) \} D_q^{j-k} \left\{ \frac{(uxq^k; q)_{\infty}}{(sxq^k; q)_{\infty}} \right\} \\ &= \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} k \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y) s^{j-k} \frac{(u/s; q)_{j-k} (uxq^j; q)_{\infty}}{(sxq^k; q)_{\infty}} \end{aligned}$$

(by using (13))

$$\square \frac{(ux; q)_{\infty}}{(sx; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y) s^{j-k} \frac{(u/s; q)_{j-k} (sx; q)_k}{(ux; q)_j}.$$

Not that, setting $u = 0$ in (19), we have the following corollary:

Corollary 2.2.1

$$\begin{aligned} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{P_n(x, y)}{(xs; q)_{\infty}} \right\} &= \frac{1}{(xs; q)_{\infty}} \\ &\times \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y) (xs; q)_k s^{j-k}, \quad |xs| < 1. \end{aligned} \tag{20}$$

3. Generating Function and its Extension for $V_n(a, b, c, f, x, y; q)$

The present section is concerned with the definition of polynomials $V_n(a, b, c, f, x, y; q)$.

The generating function and its extension for the polynomials $V_n(a, b, c, f, x, y; q)$ obtained by

utilizing the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$.

We define the polynomials $V_n(a, b, c, f, x, y; q)$ as follows:

$$V_n(a, b, c, f, x, y; q) = \sum_{k=0}^n [n]_q \frac{(a, b; q)_k}{(c; q)_k} f^k P_{n-k}(x, y). \quad (21)$$

- When $f = 0$ in equation (21), we get Cauchy polynomials $P_n(x, y)$.
- If $a = b = c = 0$ and $f = 1$ in equation (21), we obtain the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$.

Theorem 3.1 Let the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$ be defined as in (16). Then

$$\begin{aligned} & \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \{P_n(x, y)\} \\ &= V_n(a, b, c, f, x, y; q). \end{aligned} \quad (22)$$

Proof.

$$\begin{aligned} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \{P_n(x, y)\} &= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} f^k D_q^k \{P_n(x, y)\} \\ &= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} f^k \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(x, y) \quad (\text{by using (14)}) \\ &= \sum_{k=0}^n [n]_q \frac{(a, b; q)_k}{(c; q)_k} f^k P_{n-k}(x, y) \\ &= V_n(a, b, c, f, x, y; q). \end{aligned}$$

□

Theorem 3.2 (Generating function for $V_n(a, b, c, f, x, y; q)$). *Let the polynomials*

be defined as in (21). Then $V_n(a, b, c, f, x, y; q)$

$$\sum_{n=0}^{\infty} V_n(a, b, c, f, x, y; q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, ft \right), \quad |xt| < 1. \tag{23}$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} V_n(a, b, c, f, x, y; q) \frac{t^n}{(q; q)_n} &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} \right\} \\ &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} \quad (\text{by using (6)}) \\ &= (yt; q)_{\infty} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{1}{(xt; q)_{\infty}} \right\} \\ &= \\ &= \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, ft \right). \quad (\text{by using (18)}) \end{aligned}$$

- When $f = 0$ in equation (23), we recover the generating function for the Cauchy polynomials $P_n(x, y)$ (equation (6)).
- If $a = b = c = 0$ and $f = 1$ in equation (23), we have the generating function of the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ (equation (10)).

Theorem 3.3 (Extension of the generating function for $V_n(a, b, c, f, x, y; q)$). *Let*

be defined as in (21), then $V_n(a, b, c, f, x, y; q)$

$$\sum_{n=0}^{\infty} V_{n+m}(a, b, c, f, x, y; q) \frac{t^n}{(q; q)_n} = \frac{(q^m yt; q)_{\infty}}{(xt; q)_{\infty}}$$

$$\times \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q; q)_m}{(q; q)_{m-k}} P_{m-k}(x, y) (xs; q)_k t^{j-k}, \quad |xt| < 1. \quad (24)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} V_{n+m}(a, b, c, f, x, y; q) \frac{t^n}{(q; q)_n} &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \sum_{n=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n} \right\} \\ &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ P_m(x, y) \sum_{n=0}^{\infty} P_n(x, q^m y) \frac{t^n}{(q; q)_n} \right\} \\ &= (q^m y t; q)_{\infty} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{P_m(x, y)}{(xt; q)_{\infty}} \right\} \quad (\text{by using (6)}) \\ &= \square \frac{(q^m y t; q)_{\infty}}{(xt; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(a, b; q)_j}{(q, c; q)_j} f^j \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q; q)_m}{(q; q)_{m-k}} P_{m-k}(x, y) (xs; q)_k t^{j-k}. \end{aligned}$$

- Setting $m = 0$ in equation (24), we get the generating function for the polynomials

$$V_n(a, b, c, f, x, y; q).$$

- When $f = 0$ in equation (24), we get an extended generating function for the Cauchy polynomials $P_n(x, y)$

$$\sum_{n=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n} = \frac{(q^m y t; q)_{\infty}}{(xt; q)_{\infty}} P_m(x, y), \quad |xt| < 1. \quad (25)$$

- If $a = b = c = 0$ and $f = 1$ in equation (26), we obtain an extended generating function for the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$.

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} &= \frac{(q^m y t; q)_{\infty}}{(xt; q)_{\infty}} \\ &\times \sum_{j=0}^{\infty} \frac{1}{(q; q)_j} \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q; q)_m}{(q; q)_{m-k}} P_{m-k}(x, y) (xt; q)_k t^{j-k}, \quad |xt| < 1. \end{aligned} \quad (26)$$



4. Mehler’s formula and its Extension for $V_n(a, b, c, f, x, y; q)$

In this section, the operator $\mathbb{T} \begin{pmatrix} a, b \\ c \end{pmatrix}; q, fD_q$ is used to construct Mehler’s formula and its extension for the polynomials $V_n(a, b, c, f, x, y; q)$.

Theorem 4.1 (Mehler’s formula for $V_n(a, b, c, f, x, y; q)$). *Let $V_n(a, b, c, f, x, y; q)$ be defined as in (21), then*

$$\begin{aligned} & \sum_{n=0}^{\infty} V_n(a_1, b_1, c_1, f_1, x, y; q)V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \\ &= \frac{(wtx; q)_{\infty}}{(ztx; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(c_2; q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1, b_1; q)_{j+l}}{(c_1; q)_{j+l}} f_1^{j+l} \\ & \times \frac{q^{-jl}}{(q; q)_l (q; q)_j (q; q)_{k-l}} P_{k-l}(x, y) (zt)^j \frac{(q^i w/z; q)_j (ztx; q)_l}{(wtx; q)_{i+j+l}}, \end{aligned} \tag{27}$$

provided that $|ztx| < 1$.

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} V_n(a_1, b_1, c_1, f_1, x, y; q)V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} \mathbb{T} \begin{pmatrix} a_1, b_1 \\ c_1 \end{pmatrix}; q, fD_q \{P_n(x, y)\} V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \\ &= \mathbb{T} \begin{pmatrix} a_1, b_1 \\ c_1 \end{pmatrix}; q, fD_q \left\{ \sum_{n=0}^{\infty} P_n(x, y) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_2, b_2; q)_k}{(c_2; q)_k} f_2^k P_{n-k}(z, w) \frac{t^n}{(q; q)_n} \right\} \\ &= \mathbb{T} \begin{pmatrix} a_1, b_1 \\ c_1 \end{pmatrix}; q, f_1 D_q \left\{ \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} P_n(x, y) \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \frac{(a_2, b_2; q)_k}{(c; q)_k} f_2^k P_{n-k}(z, w) \frac{t^n}{(q; q)_n} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P_{n+k}(x, y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z, w) \frac{t^n}{(q; q)_n} \right\} \\
 &= \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P_k(x, y) P_n(x, q^k y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z, w) \frac{t^n}{(q; q)_n} \right\} \\
 &= \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ \sum_{k=0}^{\infty} P_k(x, y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{n=0}^{\infty} P_n(x, q^k y) P_n(z, w) \frac{t^n}{(q; q)_n} \right\} \\
 &= \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ \sum_{k=0}^{\infty} P_k(x, y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \frac{(xwt; q)_{\infty}}{(xzt; q)_{\infty}} {}_1\phi_1 \left(\begin{matrix} w/z \\ xwt \end{matrix}; q, yq^k zt \right) \right\}
 \end{aligned}$$

(by using (7))

$$\begin{aligned}
 &= \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ \sum_{k=0}^{\infty} P_k(x, y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \frac{(xwt; q)_{\infty}}{(xzt; q)_{\infty}} \right. \\
 &\quad \left. \times \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q, xwt; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i \\
 &\quad \times \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ P_k(x, y) \frac{(q^i xwt; q)_{\infty}}{(xzt; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i \frac{(q^i wtx; q)_{\infty}}{(ztx; q)_{\infty}} \\
 &\quad \times \sum_{j=0}^{\infty} \frac{(a_1, b_1; q)_j}{(q, c_1; q)_j} f_1^j \sum_{l=0}^j q^{l(l-j)} \begin{bmatrix} j \\ l \end{bmatrix} \frac{(q; q)_k}{(q; q)_{k-l}} P_{k-l}(x, y) (zt)^{j-l} \frac{(q^i w/z; q)_{j-l} (ztx; q)_l}{(q^i wtx; q)_j} \\
 &= \frac{(wtx; q)_{\infty}}{(ztx; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(c_2; q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i
 \end{aligned}$$

$$\begin{aligned} & \times \sum_{j=0}^{\infty} \frac{(a_1, b_1; q)_j}{(c_1; q)_j} f_1^j \sum_{l=0}^j \frac{q^{l(l-j)}}{(q; q)_l (q; q)_{j-l} (q; q)_{k-l}} P_{k-l}(x, y) (zt)^{j-l} \frac{(q^l w/z; q)_{j-l} (ztx; q)_l}{(wtx; q)_{i+j}} \\ & = \frac{(wtx; q)_{\infty}}{(ztx; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(c_2; q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z; q)_i}{(q; q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i \\ & \times \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1, b_1; q)_{j+l}}{(c_1; q)_{j+l}} f_1^{j+l} \frac{q^{-jl}}{(q; q)_l (q; q)_j (q; q)_{k-l}} P_{k-l}(x, y) (zt)^j \frac{(q^l w/z; q)_j (ztx; q)_l}{(wtx; q)_{i+j+l}}. \end{aligned}$$

□

• When $f_1 = f_2 = 0$ in equation (27), we get Mehler’s formula for the Cauchy polynomials

$$P_n(x, y) \text{ (equation (7))}$$

Theorem 4.2 (Extension for Mehler’s formula for $V_n(a, b, c, d, f, x, y; q)$). For $|xzt| < 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} V_{n+m}(a_1, b_1, c_1, f_1, x, y; q) V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \\ & = \frac{(wtx; q)_{\infty}}{(xzt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q; q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} yzt)^j \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_{l+i}}{(c_1; q)_{l+i}} f_1^{l+i} \\ & \times \frac{q^{-il}}{(q; q)_l (q; q)_i (q; q)_{m+k-i}} P_{m+k-i}(x, y) (zt)^l \frac{(q^j w/z; q)_l (xzt; q)_i}{(wtx; q)_{l+i+j}}. \end{aligned} \tag{28}$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} V_{n+m}(a_1, b_1, c_1, f_1, x, y; q) V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \\ & = \sum_{n=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{n+m}(x, y)\} V_n(a_2, b_2, c_2, f_2, z, w; q) \frac{t^n}{(q; q)_n} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{n+m}(x, y)\} \sum_{k=0}^n [n] \frac{(a_2, b_2; q)_k}{(c_2; q)_k} f_2^k P_{n-k}(z, w) \frac{t^n}{(q; q)_n} \\
 &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{n+m}(x, y)\} \frac{(a_2, b_2; q)_k}{(q; q)_k (q; q)_{n-k} (c_2; q)_k} f_2^k P_{n-k}(z, w) t^n \\
 &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{n+k+m}(x, y)\} \frac{(a_2, b_2; q)_k}{(q; q)_k (q; q)_n (c_2; q)_k} f_2^k P_n(z, w) t^{n+k} \\
 &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{n+k+m}(x, y)\} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z, w) \frac{t^n}{(q; q)_n} \\
 &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \{P_{m+k}(x, y) P_n(x, q^{m+k} y)\} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z, w) \frac{t^n}{(q; q)_n} \\
 &= \sum_{k=0}^{\infty} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ P_{m+k}(x, y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{n=0}^{\infty} P_n(x, q^{m+k} y) P_n(z, w) \frac{t^n}{(q; q)_n} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ P_{m+k}(x, y) \frac{(xwt; q)_{\infty}}{(xzt; q)_{\infty}} {}_1\phi_1 \left(\begin{matrix} w/z \\ xwt \end{matrix}; q, yztq^{m+k} \right) \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \\
 &\quad \times \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ P_{m+k}(x, y) \frac{(xwt; q)_{\infty}}{(xzt; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q, xwt; q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q; q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \mathbb{T} \left(\begin{matrix} a_1, b_1 \\ c_1 \end{matrix}; q, f_1 D_q \right) \left\{ P_{m+k}(x, y) \frac{(q^j xwt; q)_{\infty}}{(xzt; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q; q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \frac{(q^j wtx; q)_{\infty}}{(xzt; q)_{\infty}}
 \end{aligned}$$

$$\begin{aligned} & \times \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_l}{(q, c_1; q)_l} f_1^l \sum_{i=0}^l \begin{bmatrix} l \\ i \end{bmatrix} q^{i(i-l)} \frac{(q; q)_{m+k}}{(q; q)_{m+k-i}} P_{m+k-i}(x, y)(zt)^{l-i} \frac{(q^j w/z; q)_{l-i}(xz t; q)_i}{(q^j w t x; q)_l} \\ & = \frac{(w t x; q)_{\infty}}{(x z t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q, w t x; q)_j} (-q^{m+k} y z t)^j q^{\binom{j}{2}} \\ & \times \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_l}{(c_1; q)_l} f_1^l \sum_{i=0}^l \frac{q^{i(i-l)}}{(q; q)_{l-i}(q; q)_i} \frac{(q; q)_{m+k}}{(q; q)_{m+k-i}} P_{m+k-i}(x, y)(z t)^{l-i} \frac{(q^j w/z; q)_{l-i}(x z t; q)_i}{(w t x; q)_{l+j}} \\ & = \frac{(w t x; q)_{\infty}}{(x z t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q; q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} y z t)^j \\ & \times \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_{l+i}}{(c_1; q)_{l+i}} f_1^{l+i} \frac{q^{-il}}{(q; q)_l (q; q)_i} \frac{(q; q)_{m+k}}{(q; q)_{m+k-i}} P_{m+k-i}(x, y)(z t)^l \frac{(q^j w/z; q)_l (x z t; q)_i}{(w t x; q)_{l+i+j}}. \end{aligned}$$

□

- Setting $m = 0$ in equation (28), we get Mehler’s formula for the polynomials

$$V_n(a, b, c, f, x, y; q).$$

- When $f_1 = f_2 = 0$ in equation (28), we get an extension for Mehler’s formula for the

Cauchy polynomials $P_n(x, y)$.

$$\sum_{n=0}^{\infty} P_{n+m}(x, y) P_n(z, w) \frac{t^n}{(q; q)_n} = P_m(x, y) \frac{(x w t; q)_{\infty}}{(x z t; q)_{\infty}} {}_1\phi_1 \left(\begin{matrix} w/z \\ x w t \end{matrix}; q, q^m y z t \right), \quad |x z t| < 1.$$

- If $a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0$ and $f_1 = f_2 = 1$ in equation (28), we obtain an

extension for Mehler’s formula for bivariate Rogers-Szegö polynomials $h_n(x, y|q)$.

$$\begin{aligned} & \sum_{n=0}^{\infty} h_{n+m}(x, y) h_n(z, w) \frac{t^n}{(q; q)_n} \\ & = \frac{(w t x; q)_{\infty}}{(x z t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} t^k \sum_{j=0}^{\infty} \frac{(w/z; q)_j}{(q; q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} y z t)^j \end{aligned}$$

$$\times \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{q^{-il}}{(q; q)_i (q; q)_i (q; q)_{m+k-i}} P_{m+k-i}(x, y) (zt)^l \frac{(q^j w/z; q)_l (xzt; q)_i}{(wtx; q)_{l+i+j}}, \quad |xzt| < 1.$$

5. Rogers formula for $V_n(a, b, c, f, x, y; q)$

In this section, the operator $\mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right)$ is used to construct the Rogers formula for polynomials $V_n(a, b, c, f, x, y; q)$.

Theorem 5.1 (Rogers Formula for $V_n(a, b, c, d, f, x, y; q)$). *Let $V_n(a, b, c, d, f, x, y; q)$ be defined as in (21), then*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V_{n+m}(a, b, c, d, f, x, y; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \frac{(yt; q)_{\infty}}{(xs, xt; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt; q)_j (ys)^j}{(q, yt; q)_j} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(a, b; q)_{i+k} (xs; q)_i}{(c; q)_{i+k} (q; q)_i} (q^j ft)^i, \end{aligned} \quad (31)$$

provided that $\max\{|xs|, |xt|\} < 1$.

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V_{n+m}(a, b, c, f, x, y; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \{P_{n+m}(x, y)\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n+m}(x, y) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{(yt; q)_\infty}{(xs, xt; q)_\infty} {}_1\phi_1 \left(\begin{matrix} xt \\ yt \end{matrix}; q, ys \right) \right\} \quad (\text{by using (8)}) \\
 &= \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{(yt; q)_\infty}{(xs, xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(xt; q)_j}{(q, yt; q)_j} (-1)^j q^{\binom{j}{2}} (ys)^j \right\} \\
 &= (yt; q)_\infty \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (ys)^j}{(q, yt; q)_j} \mathbb{T} \left(\begin{matrix} a, b \\ c \end{matrix}; q, fD_q \right) \left\{ \frac{1}{(xs, xtq^j; q)_\infty} \right\} \\
 &= (yt; q)_\infty \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (ys)^j}{(q, yt; q)_j} \frac{1}{(xs, q^j xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(a, b; q)_{i+k} (xs; q)_i}{(c; q)_{i+k} (q; q)_i} (q^j ft)^i \\
 & \hspace{20em} (\text{by using (17)}) \\
 &= \frac{(yt; q)_\infty}{(xs, xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt; q)_j (ys)^j}{(q, yt; q)_j} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(a, b; q)_{i+k} (xs; q)_i}{(c; q)_{i+k} (q; q)_i} (q^j ft)^i.
 \end{aligned}$$

□

- When $f = 0$ in equation (31), we get Rogers’ formula for the Cauchy polynomials $P_n(x, y)$ (equation (8)).

- If $a = b = c = 0$ and $f = 1$ in equation (31), we obtain Rogers formula for bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ (equation (12)).

Proof. Letting $a = b = c = 0$ and $f = 1$ in equation (31), we get

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m}$$

$$\begin{aligned}
 &= \frac{(yt; q)_\infty}{(xs, xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt; q)_j (ys)^j}{(q, yt; q)_j} \sum_{k=0}^{\infty} \frac{s^k}{(q; q)_k} \sum_{i=0}^{\infty} \frac{(xs; q)_i}{(q; q)_i} (q^j t)^i \\
 &= \frac{(yt; q)_\infty}{(s, xs, xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt; q)_j (ys)^j}{(q, yt; q)_j} \sum_{i=0}^{\infty} \frac{(xs; q)_i}{(q; q)_i} (q^j t)^i \quad (\text{by using (2)}) \\
 &= \frac{(yt; q)_\infty}{(s, xs, xt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt; q)_j (ys)^j}{(q, yt; q)_j} \frac{(q^j xst; q)_\infty}{(q^j t; q)_\infty} \quad (\text{by using (1)}) \\
 &= \frac{(yt; q)_\infty}{(s, xs, xt; q)_\infty} \frac{(xst; q)_\infty}{(t; q)_\infty} \sum_{j=0}^{\infty} \frac{(t, xt; q)_j}{(q, yt, xst; q)_j} (-1)^j q^{\binom{j}{2}} (ys)^j \\
 &= \frac{(yt; q)_\infty}{(s, xs, xt; q)_\infty} \frac{(xst; q)_\infty}{(t; q)_\infty} {}_2\phi_2 \left(\begin{matrix} t, xt \\ yt, xst \end{matrix}; q, ys \right) \\
 &= \frac{(yt; q)_\infty}{(s, xs, xt; q)_\infty} \frac{(xst; q)_\infty}{(t; q)_\infty} \frac{(s; q)_\infty}{(xst; q)_\infty} {}_2\phi_1 \left(\begin{matrix} xt, y \\ yt \end{matrix}; q, s \right) \quad (\text{by using (4)}) \\
 &= \frac{(yt; q)_\infty}{(t, xs, xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} y, xt \\ yt \end{matrix}; q, s \right).
 \end{aligned}$$

Exchanging s with t , we get the required result.

□

6. Conclusions

The Cauchy polynomials $P_n(x, y)$ and the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ are special cases of the polynomials $V_n(a, b, c, f, x, y; q)$. Furthermore, the polynomials identities for $V_n(a, b, c, f, x, y; q)$ are an extension of the polynomials identities for $P_n(x, y)$ and $h_n(x, y|q)$.

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تطبيقات المؤثر T في متعددات الحدود

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المستخلص

في هذا البحث، قمنا بتعريف متعددات الحدود $V_n(a, b, c, d, f, x, y; q)$. لأيجاد الدالة المولدة، وصيغة روجرز، وصيغة ملر وتوسيعاتها لمتعددات الحدود $V_n(a, b, c, d, f, x, y; q)$ ، نستخدم المؤثر الأسّي- q $T \left(\begin{matrix} a, b \\ c \end{matrix}; q, f D_q \right)$. تم الحصول على بعض النتائج لمتعددات حدود كوشي $P_n(x, y)$ و متعددات حدود روجرز- زيكو ثنائية المتغير $h_n(x, y|q)$ عن طريق إدخال قيم خاصة في متطابقات متعددات الحدود $V_n(a, b, c, d, f, x, y; q)$.

