Applications of the Operator T in q-Polynomials

Rasha H. Jaber, Sadeq M. Khalaf, Husam L. Saad*

Department of Mathematics, College of Science, University of Basrah, Basra, Iraq

*Corresponding author E-mail: hus6274@hotmail.com

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ABSTRACT

In this paper, we define the polynomials $V_n(a,b,c,f,x,y)$. In order to determine the generating function, Rogers' formula, Mehler's formula, and their extensions for polynomials $V_n(a,b,c,f,x,y)$, we utilize the q-exponential operator T. Some results for the Cauchy polynomials $P_n(x,y)$ and the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ are obtained by inserting special values into the identities of the polynomials $V_n(a,b,c,f,x,y)$.

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1. Introduction

In this paper we will use the standard notations for basic hypergeometric series given in [1], we assume that $|q| < 1$.

Let a be a complex variable. The q-shifted factorial is defined by [1], [2]

$$(a;q)_n = \begin{cases} 1, & if \ \ n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & if \ \ n = 1,2, \dots . \end{cases}$$

We define

$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The following notation is used for the multiple q-shifted factorials:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n, \qquad n = 0,1,2, \dots \ .$$

$$(a_1, a_2, \cdots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The generalized basic hypergeometric series is defined by [1].

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r \in C$; $b_1, \dots, b_s \in C \backslash \{q^{-k}, k \in N\}$ are assumed to be such that none of the denominator factors evaluate to zero. This series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$.

The case $r = s + 1$ is the most important class of series

$${}_{s+1}\phi_s \left(\begin{matrix} a_1, \cdots, a_{s+1} \\ b_1, \cdots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_{s+1}; q)_n}{(q, b_1, \cdots, b_s; q)_n} x^n, \quad |x| < 1.$$

For $n, k \in \mathbb{N}$, the q-binomial coefficient is defined by [1]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & if\ 0 \leq k \leq n; \\ 0, & otherwise. \end{cases}$$

One of the most important identities is the Cauchy identity [1]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1}$$

Euler found the following special case of Cauchy identity [1]:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \quad |x| < 1. \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty. \tag{3}$$

Jackson's transformation formula is [1]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}\ {}_2\phi_2\left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right), \quad |z| < 1. \tag{4}$$

The Cauchy polynomials are defined by [3], [4]:

$$P_n(x,y) = (\frac{y}{x};q)_n x^n = (x-y)(x-qy)\cdots(x-q^{n-1}y), \tag{5}$$

which have the generating function [5,6,7]:

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{6}$$

Mehler's formula for $P_n(x,y)$ is [6], [8]:

$$\sum_{n=0}^{\infty} P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n} = \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}\frac{w}{z}\\ \\ xwt\end{matrix};q,yzt\right),\quad |xzt|<1. \qquad (7)$$

Rogers formula for $P_n(x,y)$ is [9]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}xt\\ yt\end{matrix};q,ys\right), \qquad (8)$$

where $\max\{|xs|,|xt|\}<1$.

The bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ is defined as :

$$h_n(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_n(x,y), \qquad (9)$$

which have the generating function [4]

$$\sum_{n=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty},\quad \max\{|t|,|xt|\}<1. \qquad (10)$$

Mehler's Formula for $h_n(x,y|q)$ is [4]

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}y,xt,\frac{v}{u}\\ \\ yt,vxt\end{matrix};q,ut\right), \qquad (11)$$

provided that $\max\{|t|,|xt|,|ut|,|uxt|\}<1$.

Roger's Formula for $h_n(x,y|q)$ is [4]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}y,xs\\ ys\end{matrix};q,t\right), \qquad (12)$$

provided that $\max\{|s|,|xs|,|xt|,|t|\}<1$.

The q- differential operator D_q is defined by [10,11,12]

$$D_q f(x) = \frac{f(x)-f(xq)}{x}.$$

The Leibniz rule for D_q is [10]:

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\{g(xq^k)\}. \tag{13}$$

The following results are easy to verify:

$$D_q^k\{P_n(x,y)\} = \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y). \tag{14}$$

$$D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{t^k}{(xt;q)_\infty}. \tag{15}$$

In 2016, inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [13] introduced the generalized q-exponential operator $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$ as follows:

$$\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right) = \sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} (fD_q)^k. \tag{16}$$

The following is how this paper is organized: In Section 2, we'll look at some identities for the q-exponential operator $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$. Section 3 presents the generating function and its extension for the polynomials $V_n(a,b,c,f,x,y)$ using the operator $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$. Section 4 establishes Mehler's formula and its extension for the polynomials $V_n(a,b,c,f,x,y)$. The operator method to Rogers' formula for polynomials $V_n(a,b,c,f,x,y)$ will be used in Section 5.

Identities for the q-Exponential Operator $\mathbb{T}$ .2

In this section, we provide some identities for the operator $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$.

Theorem 2.1 *Let the operator* $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$ *be defined as in* (16)*, then we have*

$$\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{\frac{1}{(xs,xt;q)_\infty}\right\}$$

$$= \frac{1}{(xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(fs)^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(a,b;q)_{i+k}}{(c;q)_{i+k}}\frac{(xs;q)_i}{(q;q)_i}(ft)^i, \quad \max\{|xs|,|xt|\} < 1. \qquad (17)$$

Proof.

$$\mathbb{T}\begin{pmatrix} a,b \\ c & ;q,fD_q \end{pmatrix}\left\{\frac{1}{(xt,xs;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}f^k {D_q}^k\left\{\frac{1}{(xt,xs;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}f^k\sum_{i=0}^{k}\begin{bmatrix} k \\ i \end{bmatrix}q^{i(i-k)}D_q^i\left\{\frac{1}{(xt;q)_\infty}\right\}D_q^{k-i}\left\{\frac{1}{(xsq^i;q)_\infty}\right\} \quad \text{(by using (13))}$$

$$= \sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}f^k\sum_{i=0}^{k}\begin{bmatrix} k \\ i \end{bmatrix}q^{i(i-k)}\frac{t^i}{(xt;q)_\infty}\frac{(q^i s)^{k-i}}{(xsq^i;q)_\infty}$$

$$= \frac{1}{(xt,xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(fs)^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(a,b;q)_{i+k}}{(c;q)_{i+k}}\frac{(xs;q)_i}{(q;q)_i}(ft)^i. \qquad \square$$

• Setting $s = 0$ in equation (17), we get

$$\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{1}{(xt;q)_\infty}\ {}_2\phi_1\begin{pmatrix} a,b \\ & ;q,ft \\ c \end{pmatrix}, \quad |xt| < 1. \qquad (18)$$

Theorem 2.2 *Let the operator* $\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}$ *be defined as in* (16), *then*

$$\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{P_n(x,y)\frac{(ux;q)_\infty}{(sx;q)_\infty}\right\} = \frac{(ux;q)_\infty}{(sx;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} q^{k(k-j)} \begin{bmatrix} j \\ k \end{bmatrix} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y) s^{j-k} \frac{(u/s;q)_{j-k}(sx;q)_k}{(ux;q)_j} \quad |sx| < 1. \qquad (19)$$

Proof.

$$\mathbb{T}\begin{pmatrix} a,b \\ ;q,fD_q \\ c \end{pmatrix}\left\{P_n(x,y)\frac{(ux;q)_\infty}{(sx;q)_\infty}\right\} = \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j D^j \left\{P_n(x,y)\frac{(ux;q)_\infty}{(sx;q)_\infty}\right\}$$

$$= \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} D_q^j \{P_n(x,y)\} D_q^{j-k} \left\{\frac{(uxq^k;q)_\infty}{(sxq^k;q)_\infty}\right\}$$

$$= \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} k \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y) s^{j-k} \frac{(u/s;q)_{j-k}(uxq^j;q)_\infty}{(sxq^k;q)_\infty}$$

(by using (13)) =

$$\Box \frac{(ux;q)_\infty}{(sx;q)_\infty} \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y) s^{j-k} \frac{(u/s;q)_{j-k}(sx;q)_k}{(ux;q)_j}.$$

Not that, setting $u = 0$ in (19), we have the following corollary:

Corollary 2.2.1

$$\mathbb{T}\begin{pmatrix} a,b \\ ;q,fD_q \\ c \end{pmatrix}\left\{\frac{P_n(x,y)}{(xs;q)_\infty}\right\} = \frac{1}{(xs;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y)(xs;q)_k s^{j-k}, \quad |xs| < 1. \qquad (20)$$

3. Generating Function and its Extension for $V_n(a,b,c,f,x,y;q)$

The present section is concerned with the definition of polynomials $V_n(a,b,c,f,x,y;q)$. The generating function and its extension for the polynomials $V_n(a,b,c,f,x,y;q)$ obtained by utilizing the operator $\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}$.

We define the polynomials $V_n(a,b,c,f,x,y;q)$ as follows:

$$V_n(a,b,c,f,x,y;q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(a,b;q)_k}{(c;q)_k}f^kP_{n-k}(x,y). \tag{21}$$

- When $f=0$ in equation (21), we get Cauchy polynomials $P_n(x,y)$.
- If $a=b=c=0$ and $f=1$ in equation (21), we obtain the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$.

Theorem 3.1 Let the operator $\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}$ be defined as in (16). Then

$$\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\{P_n(x,y)\}$$
$$=V_n(a,b,c,f,x,y;q). \tag{22}$$

Proof.

$$\mathbb{T}\begin{pmatrix} a,b \\ c & ;q,fD_q \end{pmatrix}\{P_n(x,y)\}=\textstyle\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}f^kD_q^k\{P_n(x,y)\}$$

$$=\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}f^k\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y)\quad \text{(by using (14))}$$

$$=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(a,b;q)_k}{(c;q)_k}f^kP_{n-k}(x,y)$$

$$=V_n(a,b,c,f,x,y;q).$$

□

Theorem 3.2 (Generating function for $V_n(a,b,c,f,x,y;q)$). *Let the polynomials be defined as in* (21). *Then* $V_n(a,b,c,f,x,y;q)$

$$\sum_{n=0}^{\infty} V_n(a,b,c,f,x,y;q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,ft\right), \quad |xt|<1. \tag{23}$$

Proof.

$$\sum_{n=0}^{\infty} V_n(a,b,c,f,x,y;q)\frac{t^n}{(q;q)_n} = \mathbb{T}\left(\begin{matrix}a,b\\ c\end{matrix};q,fD_q\right)\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{T}\left(\begin{matrix}a,b\\ c\end{matrix};q,fD_q\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} \quad \text{(by using (6))}$$

$$= (yt;q)_\infty \mathbb{T}\left(\begin{matrix}a,b\\ c\end{matrix};q,fD_q\right)\left\{\frac{1}{(xt;q)_\infty}\right\}$$

$$= \frac{(yt;q)_\infty}{(xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,ft\right). \quad \text{(by using (18))}$$

□

- When $f=0$ in equation (23), we recover the generating function for the Cauchy polynomials $P_n(x,y)$ (equation (6)).
- If $a=b=c=0$ and $f=1$ in equation (23), we have the generating function of the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ (equation (10)).

Theorem 3.3 (Extension of the generating function for $V_n(a,b,c,f,x,y;q)$). *Let be defined as in* (21), *then* $V_n(a,b,c,f,x,y;q)$

$$\sum_{n=0}^{\infty} V_{n+m}(a,b,c,f,x,y;q)\frac{t^n}{(q;q)_n} = \frac{(q^m yt;q)_\infty}{(xt;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q;q)_m}{(q;q)_{m-k}} P_{m-k}(x,y)(xs;q)_k t^{j-k}, \quad |xt| < 1. \tag{24}$$

Proof.

$$\sum_{n=0}^{\infty} V_{n+m}(a,b,c,f,x,y;q) \frac{t^n}{(q;q)_n} = \mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix} \left\{ \sum_{n=0}^{\infty} P_{n+m}(x,y) \frac{t^n}{(q;q)_n} \right\}$$

$$= \mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix} \left\{ P_m(x,y) \sum_{n=0}^{\infty} P_n(x,q^m y) \frac{t^n}{(q;q)_n} \right\}$$

$$= (q^m yt;q)_\infty \mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix} \left\{ \frac{P_m(x,y)}{(xt;q)_\infty} \right\} \ (by \ using \ (6))$$

$$= \frac{(q^m yt;q)_\infty}{(xt;q)_\infty} \sum_{j=0}^{\infty} \frac{(a,b;q)_j}{(q,c;q)_j} f^j \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q;q)_m}{(q;q)_{m-k}} P_{m-k}(x,y)(xs;q)_k t^{j-k}.$$ □

- Setting $m = 0$ in equation (24), we get the generating function for the polynomials $V_n(a,b,c,f,x,y;q)$.
- When $f = 0$ in equation (24), we get an extended generating function for the Cauchy polynomials $P_n(x,y)$

$$\sum_{n=0}^{\infty} P_{n+m}(x,y) \frac{t^n}{(q;q)_n} = \frac{(q^m yt;q)_\infty}{(xt;q)_\infty} P_m(x,y), \quad |xt| < 1. \tag{25}$$

- If $a = b = c = 0$ and $f = 1$ in equation (26), we obtain an extended generating function for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$.

$$\sum_{n=0}^{\infty} h_{n+m}(x,y|q) \frac{t^n}{(q;q)_n} = \frac{(q^m yt;q)_\infty}{(xt;q)_\infty}$$

$$\times \sum_{j=0}^{\infty} \frac{1}{(q;q)_j} \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} q^{k(k-j)} \frac{(q;q)_m}{(q;q)_{m-k}} P_{m-k}(x,y)(xt;q)_k t^{j-k}, \quad |xt| < 1. \tag{26}$$

4. Mehler's formula and its Extension for $V_n(a,b,c,f,x,y;q)$

In this section, the operator $\mathbb{T}\left(\begin{matrix}a,b\\ c\end{matrix};q,fD_q\right)$ is used to construct Mehler's formula and its extension for the polynomials $V_n(a,b,c,f,x,y;q)$.

Theorem 4.1 (Mehler's formula for $V_n(a,b,c,f,x,y;q)$). *Let* $V_n(a,b,c,f,x,y;q)$ *be defined as in* (21), *then*

$$\sum_{n=0}^{\infty} V_n(a_1,b_1,c_1,f_1,x,y;q)V_n(a_2,b_2,c_2,f_2,z,w;q)\frac{t^n}{(q;q)_n}$$

$$=\frac{(wtx;q)_\infty}{(ztx;q)_\infty}\sum_{k=0}^{\infty}\frac{(a_2,b_2;q)_k}{(c_2;q)_k}(f_2t)^k\sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}(yztq^k)^i\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_1,b_1;q)_{j+l}}{(c_1;q)_{j+l}}f_1^{j+l}$$

$$\times\frac{q^{-jl}}{(q;q)_l(q;q)_j(q;q)_{k-l}}P_{k-l}(x,y)(zt)^j\frac{(q^iw/z;q)_j(ztx;q)_l}{(wtx;q)_{i+j+l}}, \tag{27}$$

provided that $|ztx|<1$.

Proof.

$$\sum_{n=0}^{\infty} V_n(a_1,b_1,c_1,f_1,x,y;q)V_n(a_2,b_2,c_2,f_2,z,w;q)\frac{t^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\mathbb{T}\left(\begin{matrix}a_1,b_1\\ c_1\end{matrix};q,fD_q\right)\{P_n(x,y)\}V_n(a_2,b_2,c_2,f_2,z,w;q)\frac{t^n}{(q;q)_n}$$

$$=\mathbb{T}\left(\begin{matrix}a_1,b_1\\ c_1\end{matrix};q,fD_q\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a_2,b_2;q)_k}{(c_2;q)_k}f_2^kP_{n-k}(z,w)\frac{t^n}{(q;q)_n}\right\}$$

$$=\mathbb{T}\left(\begin{matrix}a_1,b_1\\ ,\\ c_1\end{matrix};q,f_1D_q\right)\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}P_n(x,y)\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\frac{(a_2,b_2;q)_k}{(c;q)_k}f_2^kP_{n-k}(z,w)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ , \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P_{n+k}(x,y)\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k P_n(z,w)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P_k(x,y)P_n(x,q^k y)\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k P_n(z,w)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{\sum_{k=0}^{\infty} P_k(x,y)\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k \sum_{n=0}^{\infty} P_n(x,q^k y)P_n(z,w)\frac{t^n}{(q;q)_n}\right\}$$

$$= \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{\sum_{k=0}^{\infty} P_k(x,y)\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\ {}_1\phi_1\left(\begin{matrix} w/z \\ \\ xwt \end{matrix}; q, yq^k zt\right)\right\}$$

(by using (7))

$$= \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{\sum_{k=0}^{\infty} P_k(x,y)\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(ft_2)^k \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\right.$$

$$\left.\times\sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q,xwt;q)_i}(-1)^i q^{\binom{i}{2}}(yztq^k)^i\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k \sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}(yztq^k)^i$$

$$\times\, \mathbb{T}\left(\begin{matrix} a_1, b_1 \\ \\ c_1 \end{matrix}; q, f_1 D_q\right)\left\{P_k(x,y)\frac{(q^i xwt;q)_\infty}{(xzt;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(a_2,b_2;q)_k}{(q,c_2;q)_k}(f_2 t)^k \sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}(yztq^k)^i \frac{(q^i wtx;q)_\infty}{(ztx;q)_\infty}$$

$$\times\sum_{j=0}^{\infty}\frac{(a_1,b_1;q)_j}{(q,c_1;q)_j}f_1^j \sum_{l=0}^{j} q^{l(l-j)}\begin{bmatrix} j \\ l \end{bmatrix}\frac{(q;q)_k}{(q;q)_{k-l}}P_{k-l}(x,y)(zt)^{j-l}\frac{(q^i w/z;q)_{j-l}(ztx;q)_l}{(q^i wtx;q)_j}$$

$$= \frac{(wtx;q)_\infty}{(ztx;q)_\infty}\sum_{k=0}^{\infty}\frac{(a_2,b_2;q)_k}{(c_2;q)_k}(f_2 t)^k \sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}(yztq^k)^i$$

$$\times \sum_{j=0}^{\infty} \frac{(a_1,b_1;q)_j}{(c_1;q)_j} f_1^j \sum_{l=0}^{j} \frac{q^{l(l-j)}}{(q;q)_l(q;q)_{j-l}(q;q)_{k-l}} P_{k-l}(x,y)(zt)^{j-l} \frac{(q^i w/z;q)_{j-l}(ztx;q)_l}{(wtx;q)_{i+j}}$$

$$= \frac{(wtx;q)_\infty}{(ztx;q)_\infty} \sum_{k=0}^{\infty} \frac{(a_2,b_2;q)_k}{(c_2;q)_k} (f_2 t)^k \sum_{i=0}^{\infty} \frac{(w/z;q)_i}{(q;q)_i} (-1)^i q^{\binom{i}{2}} (yztq^k)^i$$

$$\times \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1,b_1;q)_{j+l}}{(c_1;q)_{j+l}} f_1^{j+l} \frac{q^{-jl}}{(q;q)_l(q;q)_j(q;q)_{k-l}} P_{k-l}(x,y)(zt)^j \frac{(q^i w/z;q)_j(ztx;q)_l}{(wtx;q)_{i+j+l}}.$$

□

• When $f_1 = f_2 = 0$ in equation (27), we get Mehler's formula for the Cauchy polynomials

$P_n(x,y)$ (equation (7))

Theorem 4.2 (Extension for Mehler's formula for $V_n(a,b,c,d,f,x,y;q)$). *For* $|xzt| < 1$, *we have*

$$\sum_{n=0}^{\infty} V_{n+m}(a_1,b_1,c_1,f_1,x,y;q) V_n(a_2,b_2,c_2,f_2,z,w;q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(wtx;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a_2,b_2;q)_k}{(q,c_2;q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q;q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} yzt)^j \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_1,b_1;q)_{l+i}}{(c_1;q)_{l+i}} f_1^{l+i}$$

$$\times \frac{q^{-il}}{(q;q)_l(q;q)_i} \frac{(q;q)_{m+k}}{(q;q)_{m+k-i}} P_{m+k-i}(x,y)(zt)^l \frac{(q^j w/z;q)_l(xzt;q)_i}{(wtx;q)_{l+i+j}}. \tag{28}$$

Proof.

$$\sum_{n=0}^{\infty} V_{n+m}(a_1,b_1,c_1,f_1,x,y;q) V_n(a_2,b_2,c_2,f_2,z,w;q) \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 & \\ & ; q, f_1 D_q \\ c_1 & \end{pmatrix} \{P_{n+m}(x,y)\} V_n(a_2,b_2,c_2,f_2,z,w;q) \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \{P_{n+m}(x,y)\} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_2, b_2; q)_k}{(c_2; q)_k} f_2^k P_{n-k}(z,w) \frac{t^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \{P_{n+m}(x,y)\} \frac{(a_2, b_2; q)_k}{(q;q)_k (q;q)_{n-k} (c_2;q)_k} f_2^k P_{n-k}(z,w) t^n$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \{P_{n+k+m}(x,y)\} \frac{(a_2, b_2; q)_k}{(q;q)_k (q;q)_n (c_2;q)_k} f_2^k P_n(z,w) t^{n+k}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \{P_{n+k+m}(x,y)\} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z,w) \frac{t^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \{P_{m+k}(x,y) P_n(x, q^{m+k} y)\} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k P_n(z,w) \frac{t^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \left\{ P_{m+k}(x,y) \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{n=0}^{\infty} P_n(x, q^{m+k} y) P_n(z,w) \frac{t^n}{(q;q)_n} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \left\{ P_{m+k}(x,y) \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \; {}_1\phi_1 \begin{pmatrix} w/z \\ & ; q, yztq^{m+k} \\ xwt \end{pmatrix} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k$$

$$\times \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \left\{ P_{m+k}(x,y) \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q, xwt; q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q;q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \mathbb{T}\begin{pmatrix} a_1, b_1 \\ & ; q, f_1 D_q \\ c_1 \end{pmatrix} \left\{ P_{m+k}(x,y) \frac{(q^j xwt;q)_\infty}{(xzt;q)_\infty} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q;q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}} \frac{(q^j wtx;q)_\infty}{(xzt;q)_\infty}$$

$$\times \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_l}{(q, c_1; q)_l} f_1^l \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} q^{i(i-l)} \frac{(q;q)_{m+k}}{(q;q)_{m+k-i}} P_{m+k-i}(x,y)(zt)^{l-i} \frac{(q^j w/z;q)_{l-i}(xzt;q)_i}{(q^j wtx;q)_l}$$

$$= \frac{(wtx;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q, wtx;q)_j} (-q^{m+k} yzt)^j q^{\binom{j}{2}}$$

$$\times \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_l}{(c_1; q)_l} f_1^l \sum_{i=0}^{l} \frac{q^{i(i-l)}}{(q;q)_{l-i}(q;q)_i} \frac{(q;q)_{m+k}}{(q;q)_{m+k-i}} P_{m+k-i}(x,y)(zt)^{l-i} \frac{(q^j w/z;q)_{l-i}(xzt;q)_i}{(wtx;q)_{l+j}}$$

$$= \frac{(wtx;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a_2, b_2; q)_k}{(q, c_2; q)_k} (f_2 t)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q;q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} yzt)^j$$

$$\times \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_1, b_1; q)_{l+i}}{(c_1; q)_{l+i}} f_1^{l+i} \frac{q^{-il}}{(q;q)_l (q;q)_i} \frac{(q;q)_{m+k}}{(q;q)_{m+k-i}} P_{m+k-i}(x,y)(zt)^l \frac{(q^j w/z;q)_l (xzt;q)_i}{(wtx;q)_{l+i+j}}.$$

□

- Setting $m = 0$ in equation (28), we get Mehler's formula for the polynomials $V_n(a,b,c,f,x,y;q)$.
- When $f_1 = f_2 = 0$ in equation (28), we get an extension for Mehler's formula for the Cauchy polynomials $P_n(x,y)$.

$$\sum_{n=0}^{\infty} P_{n+m}(x,y) P_n(z,w) \frac{t^n}{(q;q)_n} = P_m(x,y) \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \, {}_1\phi_1 \left(\begin{matrix} w/z \\ xwt \end{matrix} ; q, q^m yzt \right), \quad |xzt| < 1.$$

- If $a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0$ and $f_1 = f_2 = 1$ in equation (28), we obtain an extension for Mehler's formula for bivariate Rogers-Szegö polynomials $h_n(x,y|q)$.

$$\sum_{n=0}^{\infty} h_{n+m}(x,y) h_n(z,w) \frac{t^n}{(q;q)_n}$$

$$= \frac{(wtx;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} t^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q;q)_j} (-1)^j q^{\binom{j}{2}} (q^{m+k} yzt)^j$$

$$\times \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{q^{-il}}{(q;q)_l (q;q)_i} \frac{(q;q)_{m+k}}{(q;q)_{m+k-i}} P_{m+k-i}(x,y)(zt)^l \frac{(q^j w/z;q)_l (xzt;q)_i}{(wtx;q)_{l+i+j}}, \quad |xzt| < 1.$$

5. Rogers formula for $V_n(a,b,c,f,x,y;q)$

In this section, the operator $\mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right)$ is used to construct the Rogers formula for polynomials $V_n(a,b,c,f,x,y;q)$.

Theorem 5.1 (Rogers Formula for $V_n(a,b,c,d,f,x,y;q)$). *Let* $V_n(a,b,c,d,f,x,y;q)$ *be defined as in* (21), *then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V_{n+m}(a,b,c,d,f,x,y;q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \frac{(yt;q)_\infty}{(xs,xt;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (xt;q)_j (ys)^j}{(q,yt;q)_j} \sum_{k=0}^{\infty} \frac{(fs)^k}{(q;q)_k} \sum_{i=0}^{\infty} \frac{(a,b;q)_{i+k}}{(c;q)_{i+k}} \frac{(xs;q)_i}{(q;q)_i} (q^j ft)^i, \qquad (31)$$

provided that $\max\{|xs|,|xt|\} < 1$.

Proof.

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V_{n+m}(a,b,c,f,x,y;q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right) \{P_{n+m}(x,y)\} \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \mathbb{T}\left(\begin{matrix} a,b \\ c \end{matrix}; q, fD_q\right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \right\}$$

$$= \mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{\frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\ {}_1\phi_1\begin{pmatrix} xt \\ & ;q,ys \\ yt \end{pmatrix}\right\} \quad (\ by\ \ using\ \ (8)\ )$$

$$= \mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{\frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(xt;q)_j}{(q,yt;q)_j}(-1)^j q^{\binom{j}{2}}(ys)^j\right\}$$

$$= (yt;q)_\infty \sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(ys)^j}{(q,yt;q)_j}\mathbb{T}\begin{pmatrix} a,b \\ & ;q,fD_q \\ c \end{pmatrix}\left\{\frac{1}{(xs,xtq^j;q)_\infty}\right\}$$

$$= (yt;q)_\infty \sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(ys)^j}{(q,yt;q)_j}\frac{1}{(xs,q^jxt;q)_\infty}\sum_{k=0}^{\infty}\frac{(fs)^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(a,b;q)_{i+k}}{(c;q)_{i+k}}\frac{(xs;q)_i}{(q;q)_i}(q^jft)^i$$

(by using (17))

$$= \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(xt;q)_j(ys)^j}{(q,yt;q)_j}\sum_{k=0}^{\infty}\frac{(fs)^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(a,b;q)_{i+k}}{(c;q)_{i+k}}\frac{(xs;q)_i}{(q;q)_i}(q^jft)^i.$$

□

- When $f = 0$ in equation (31), we get Rogers' formula for the Cauchy polynomials $P_n(x,y)$ (equation (8)).

- If $a = b = c = 0$ and $f = 1$ in equation (31), we obtain Rogers formula for bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ (equation (12)).

Proof. Letting $a = b = c = 0$ and $f = 1$ in equation (31), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$= \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(xt;q)_j(ys)^j}{(q,yt;q)_j}\sum_{k=0}^{\infty}\frac{s^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(xs;q)_i}{(q;q)_i}(q^jt)^i$$

$$= \frac{(yt;q)_\infty}{(s,xs,xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(xt;q)_j(ys)^j}{(q,yt;q)_j}\sum_{i=0}^{\infty}\frac{(xs;q)_i}{(q;q)_i}(q^jt)^i \quad (\, by\ \ using\ \ (2)\,)$$

$$= \frac{(yt;q)_\infty}{(s,xs,xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(xt;q)_j(ys)^j}{(q,yt;q)_j}\frac{(q^jxst;q)_\infty}{(q^jt;q)_\infty} \quad (\, by\ \ using\ \ (1)\,)$$

$$= \frac{(yt;q)_\infty}{(s,xs,xt;q)_\infty}\frac{(xst;q)_\infty}{(t;q)_\infty}\sum_{j=0}^{\infty}\frac{(t,xt;q)_j}{(q,yt,xst;q)_j}(-1)^j q^{\binom{j}{2}}(ys)^j$$

$$= \frac{(yt;q)_\infty}{(s,xs,xt;q)_\infty}\frac{(xts;q)_\infty}{(t;q)_\infty}\ {}_2\phi_2\left(\begin{matrix} t,xt \\ \\ yt,sxt\end{matrix};q,ys\right)$$

$$= \frac{(yt;q)_\infty}{(s,xs,xt;q)_\infty}\frac{(xts;q)_\infty}{(t;q)_\infty}\frac{(s;q)_\infty}{(xts;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} xt,y \\ \\ yt\end{matrix};q,s\right) \quad (\, by\ \ using\ \ (4)\,)$$

$$= \frac{(yt;q)_\infty}{(t,xs,xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y,xt \\ \\ yt\end{matrix};q,s\right).$$

Exchanging s with t, we get the required result.

□

6. Conclusions

The Cauchy polynomials $P_n(x,y)$ and the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ are special cases of the polynomials $V_n(a,b,c,f,x,y;q)$. Furthermore, the polynomials identities for $V_n(a,b,c,f,x,y;q)$ are an extension of the polynomials identities for $P_n(x,y)$ and $h_n(x,y|q)$.

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تطبيقات المؤثر T في متعددات الحدود

رشا هادي جابر صادق ماجد خلف حسام لوتي سعد

قسم الرياضيات, كلية العلوم, جامعة البصرة, البصرة, العراق

المستخلص

في هذا البحث، قمنا بتعريف متعددات الحدود $V_n(a,b,c,d,f,x,y;q)$. لأيجاد الدالة المولدة، وصيغة روجرز، وصيغة ملر وتوسيعاتها لمتعددات الحدود $V_n(a,b,c,d,f,x,y;q)$، نستخدم المؤثر الأسي-q $\mathbb{T}\begin{pmatrix} a,b \\ c \end{pmatrix}; q, fD_q\end{pmatrix}$. تم الحصول على بعض النتائج لمتعددات حدود كوشي $P_n(x,y)$ ومتعددات حدود روجرز- زيكو ثنائية المتغير $h_n(x,y|q)$ عن طريق إدخال قيم خاصة في متطابقات متعددات الحدود $V_n(a,b,c,d,f,x,y;q)$.