Approximate Solution of Quadratic Time Varying Optimal Control Problems via Differential Transform Method

Sanaa L. Khalaf , Khulood K. Kassid

Department of Mathematics, College of Sciences, University of Basrah, Basra, Iraq

*Corresponding author: **sanaasanaa1978@yahoo.com**

Doi 10.29072/basjs.20200103

Abstract:

Because of there is no general analytical method for solving the quadratic time varying optimal control problems (QTVOCPs), many authors adopt the numerical methods for this purpose. This motivates some authors to use the differential transform method (DTM) for finding an approximate solution for the linear quadratic optimal control problems (LQOCPs) and a class of nonlinear quadratic optimal control. In this paper, we propose an algorithm depend on applying DTM about the two end points of the time horizon to find an approximate solution for the linear or nonlinear QTVOCPs. To show the reliability, accuracy and efficiency of suggested method, some illustrated examples have been provided.

Article inf.

Received:

2/1/2020

Accepted

11/3/2020

Published

30/4/2020

Keywords

Quadratic optimal control, Differential transforms method

1. Introduction

Over the last decades, the studying the optimal control theory is very important since it has provided a powerful tool for the solving a wide spectrum of real life problems. Therefore, the optimal control strategies are fundamental in many fields of applied science like aerospace engineering, engineering, economics, chemistry, physics and biology [1, 2, 3, 4].

The general formula of optimal control problem (OCP) consists two main parts: performance index (cost function) and state space equations (dynamic system). The optimal control problem can be summarized by the process of finding a control variables and state variables such that satisfy the state space equations over a period of time to minimize (maximize) the given performance index. Based on the time domain (discrete, continuous) of performance index and the types of constraints the OCP is classified into various types.

In fact, the historical roots of optimal control go back to the calculus of variations , which is stretching back over 380 years ago, where OCPs was considered as applications and extensions of the calculus of variations [5, 6, 7, 8]. Anyway, the Cold War had due the primary of the evolution of optimal control theory in two different directions: the dynamic programming and principle of Pontryagin. The dynamic programming was introduced by American mathematician, R. E. Bellman, in 1953 while the principle of Pontryagin was introduced by some of Russian mathematician supervised by L. Pontryagin in 1956 [9].

There are two main branches to solve the OCPs: direct [10] or indirect methods. The indirect methods depend on solving the Pontryagin's system by using suitable numerical method like shooting method [11], finite difference methods [12]. The direct methods based on discretization the time horizon of the control problem and use the nonlinear programming techniques to the resulting problem [13].

More recently, the linear quadratic OCPs has been solved by using some semi-analytical methods for instance, the homotopy perturbation [14], the Adomian decomposition [15], the variational iteration [16], and the differential transformation [17, 18]. In this paper, we will introduce an algorithm to solve the linear or nonlinear QTVOCPs. We compare the suggested method with the methods in [17, 18]. The results show the reliability, accuracy and efficiency of the suggested method.

1- Problem Formulation

In many real life applications, the issues of computing certain variables (control variables) in order to optimize a specific performance functional (cost functional) and that achieves set of differential equations (state space equations). These issues are called optimal control problems (OCPs). In fact, there are many different kinds of OCPs depending on the types of the cost functions, the types of the state space equations, and admissible set control variables. Consider the dynamical system

$$\frac{dx}{dt} = G_1(t, x(t)) + G_2(t, x(t))u(t), \qquad x(a) = x_a \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is called the state variable and satisfy eq.(1) for a given control input $u(t) \in \mathbf{R}^m$. The performance functional is

$$\text{Min } \Im[x(t), u(t)] = \frac{1}{2} x^T(b)\, \mathrm{M}\, x(b) + \int_a^b F(x(s), u(s))\, ds \tag{2}$$

where M is positive definite $n \times n$ matrix, $F(x,u)$ is convex function, a and b denoted to the initial time and the terminal time respectively. The OCP is how can find a control $u^*(t) \in \mathbf{R}^m$ that it satisfies eq.(1) and gives the minimum value for eq.(2). In order to find $u^*(t) \in \mathbf{R}^m$, the optimal control theory [19] will be used. So the first step is constructing the Hamiltonian function as follows:

$$\mathrm{H}(t, x, \lambda, u) = F(x, u) + \lambda^T [\mathrm{G}_1(t, x) + G_2(t, x)u] \tag{3}$$

where $\lambda(t) \in \mathbf{R}^n$ is the co-state variable. The second step is applying the following Pontryagin's minimum principle [19]:

$$\frac{dx^*}{dt} = \mathrm{H}_{\lambda^*}(t, x^*, \lambda^*, u^*) \tag{4}$$

$$\frac{d\lambda^*}{dt} = -\mathrm{H}_{x^*}(t, x^*, \lambda^*, u^*) \tag{5}$$

$$\mathrm{H}_{u^*}(t, x^*, \lambda^*, u^*) = 0 \tag{6}$$

$$\mathrm{H}(t, x^*, \lambda^*, u) \geq \mathrm{H}(t, x^*, \lambda^*, u^*) \tag{7}$$

$$\lambda(b) = \mathrm{M}\, x(b) \tag{8}$$

The optimal $u^*(t) \in \mathbf{R}^m$ must satisfy eq.(6) and eq.(7). If we can find $u^*(t)$ in term of $x^*(t)$ and/or $\lambda^*(t)$ and substituted in eq.(4) and eq.(5), the OCP equivalent to solve eq.(4) and eq.(5) with $x(a) = x_a$ and $\lambda(b) = \mathrm{M}\, x(b)$.

If $\mathrm{A}_{n\times n}$ is positive semidefinite and $\mathrm{B}_{m\times m}$ is positive definite matrices, such that F in eq.(2) is given by $F(x(t), u(t)) = \frac{1}{2} x^T(t)\, \mathrm{A}_{n\times n}\, x(t) + \frac{1}{2} u^T(t)\, \mathrm{B}_{m\times m}\, u(t)$, then the OCP under consideration is called nonlinear QTVOCPs. In additional, if G_1 and G_2 in eq.(1) are constant matrices, then the OCP under consideration is called linear quadratic optimal control Problems (LQOCPs).

2- Differential Transform Method (DTM)

Inspired of Taylor series expansion, Zhou [20] introduced the DTM to solve electrical circuit problems in linear or nonlinear case. After that, the DTM received great attention from researchers and it was used to solve many types of problems in various fields. In fact, this method does not require any linearization or a domain discretization, so it is not affected by discretization errors. In general, the DTM converts the given differential equations to a system of algebraic equations. In this section, we will review the main properties and theorem related to DTM.

Let $x(t):\Omega\subseteq\mathbf{R}\to\mathbf{R}$ be smooth function in the domain Ω and let $a\in\Omega$, then the Taylor series of $x(t)$ about $t=a$ is given by

$$x(t)=\sum_{s=0}^{\infty}\frac{(t-a)^s}{s!}\frac{d^s x(t)}{dt^s}\bigg|_{t=a}, \quad \forall t\in\Omega \tag{9}$$

Let

$$\mathrm{X[s]}=\frac{1}{\mathrm{s}!}\frac{d^s x(t)}{dt^s}\bigg|_{t=a} \tag{10}$$

then eq.(9) is converted to

$$x(t)=\sum_{s=0}^{\infty}\mathrm{X[s]}(t-a)^s, \quad \forall t\in\Omega \tag{11}$$

Now, $\mathrm{X[s]}$ in eq.(10) is the differential transform (DT) about $t=a$ of the k^{th} derivative of $x(t)$, while $x(t)$ in eq.(11) is the differential inverse transform (DIT) about $t=a$ of $\mathrm{X[s]}$ [21]. It seems that the DTM is same as a Taylor series method, but the DTM does not compute any derivatives symbolically. In fact, the relative derivatives are evaluated by an iterative procedure. On the other hand, it is easily proven the following theorems which are related to the essential operations of the DTM.

Theorem 1 [20] If $x(t)=\gamma_1 x_1(t)\pm\gamma_2 x_2(t)$ then the DT of $x(t)$ about $t=a$ is $\mathrm{X[s]}=\gamma_1\mathrm{X}_1[s]\pm\gamma_2\mathrm{X}_2[s],\ \forall\,\mathrm{s}=0,1,2,\dots$, where γ_1 and γ_2 are real constant.

Theorem 2 [20] If $x_3(\mathrm{t})=x_1(\mathrm{t})x_2(\mathrm{t})$ then the DT of $x_3(t)$ about $t=a$ is

$$\mathrm{X}_3[s]=\sum_{j=0}^{s}\mathrm{X}_1[j]\mathrm{X}_2[s-j], \quad \forall\,\mathrm{s}=0,1,2,\dots$$

Theorem 3 [20] If $x_2(\mathrm{t}) = \dfrac{d^n x_1(\mathrm{t})}{dt^n}$ then the DT of $x_2(t)$ about $t = a$ is

$\mathrm{X}_2[s] = \dfrac{(n+s)!}{s!}\mathrm{X}_1[s], \quad \forall\, \mathrm{s} = 0,1,2,\dots$.

3- **Main Result**

In this section, we propose an algorithm to find an optimal control $u^*(t):[a,b] \to \mathbf{R}$ that minimize the following performance function

$$\text{Min } \Im[x(t),u(t)] = \frac{1}{2}x^T(b)\,\mathrm{M}\,x(b) + \frac{1}{2}\int_a^b [x^T(\mathrm{s})\,\mathrm{A}_{n\times n}\,x(\mathrm{s}) + u^T(\mathrm{s})\,\mathrm{B}_{m\times m}\,u(\mathrm{s})]ds \tag{12}$$

and satisfy the following dynamical system

$$\frac{dx}{dt} = \mathrm{C}_{n\times n}(t,x) + \mathrm{D}_{n\times m}(t,x)\,u(t), \qquad x(a) = x_a \tag{13}$$

where $\mathrm{A}_{n\times n}$ is positive semidefinite, $\mathrm{B}_{m\times m}$ and $\mathrm{M}_{n\times n}$ are positive definite matrices.

Now, we construct the Hamiltonian function as follows:

$$\mathrm{H}(t,x,\lambda,u) = \frac{1}{2}[x^T(t)\,\mathrm{A}_{n\times n}\,x(t) + u^T(t)\,\mathrm{B}_{m\times m}\,u(t)] + \lambda^T[\mathrm{C}_{n\times n}(t,x(t)) + \mathrm{D}_{n\times m}(t,x(t))\,u(t)] \tag{14}$$

By applying the Pontryagin's minimum principle, we have

$$\frac{dx^*}{dt} = \mathrm{C}(t,x^*(t)) + \mathrm{D}(t,x^*(t))u^*(t), \qquad x^*(a) = x_a \tag{15}$$

$$\frac{d\lambda^*}{dt} = -\mathrm{A}\,x^*(t) - \left(\frac{\partial \mathrm{C}^T(t,x)}{\partial x} + \frac{\partial D^T(t,x)}{\partial x}\right)_{x(t)=x^*(t)} \lambda^*(\mathrm{t}), \qquad \lambda(b) = \mathrm{M}\,x(b) \tag{16}$$

$$\mathrm{H}_{u^*}(t,x^*,\lambda^*,u^*) = \mathrm{B}u^*(t) + \mathrm{D}^T(t,x^*(t))\lambda^*(t) = 0 \tag{17}$$

Since B is positive definite it is invertible and hence we can find the optimal control $u^*(t)$ in the form

$$u^*(t) = -\mathrm{B}^{-1}\,\mathrm{D}^T(t,x^*(t))\lambda^*(\mathrm{t}) \tag{18}$$

It is clear that the optimal control $u^*(t)$ in eq.(18) depend on the co-state $\lambda^*(t)$. So, we try to find $\lambda^*(t)$ by substituting eq.(18) in the state dynamic system eq.(15) to get

$$\frac{dx^*}{dt} = \mathrm{C}(t, x^*(t)) - \mathrm{D}(t, x^*(t))\,\mathrm{B}^{-1}\,\mathrm{D}^T(t, x^*(t))\lambda^*(t), \qquad x(a) = x_a. \tag{19}$$

Then, we attempt to solve the system, eq.(19) and eq.(16), of two point boundary value problem (TPBVP) to find $\lambda^*(t)$ and $x^*(t)$ for all $t \in [a,b]$. In other words, the solution of QTVOCP is equivalent to solve the TPBVP eq.(19) and eq.(16) and then use eq.(18) to find the optimal control $u^*(t)$. For this purpose the following algorithms are investigate based on the DTM.

If all the matrices in eq.(13) are constant, we review the method in [17] and rewrite following algorithm to find the optimal solution of LQOCP.

Algorithm 1

1- Find the exponential matrix $\Psi(t,\mathrm{a})$ for the state eq.(19) and co-state equations eq.(16), that is

$$\begin{pmatrix} x^*(t) \\ \lambda^*(t) \end{pmatrix} = \Psi(t,a)\begin{pmatrix} x^*(a) \\ \lambda^*(a) \end{pmatrix} \tag{20}$$

2- Substitute $t = b$ in eq.(20), we have

$$\begin{pmatrix} x^*(b) \\ \lambda^*(b) \end{pmatrix} = \Psi(b,a)\begin{pmatrix} x^*(a) \\ \lambda^*(a) \end{pmatrix} \tag{21}$$

3- Solve the linear eq.(21) for $\lambda^*(a)$ and $x^*(b)$ in term of $x^*(a)$ and $\lambda^*(b)$.

4- Apply the DTM about a to the state eq.(19) and co-state equations eq.(16), we have an iterative relationship.

5- Apply the DTM about a for the $x^*(a) = x_a$ and $\lambda^*(a)$, we have $X[0] = x_a$ and $\Lambda[0] = \lambda^*(a)$.

6- Carry out the iterative relationship in step (4) with the result in step (5), we have $X[s]$ and $\Lambda[s]$ for all $s = 0,1,2,\ldots,N$.

7- To find $x^*(t)$ and $\lambda^*(t)$, use the result in step(4) for $x^*(t) \simeq \sum_{s=0}^{N} X[\mathrm{s}](t-a)^{\mathrm{s}}$ and $\lambda^*(t) \simeq \sum_{s=0}^{N} \Lambda[\mathrm{s}](t-a)^{\mathrm{s}}$.

8- By using the result in step (7) and by applying eq.(18), the approximate solution of optimal control is

$$u^*(t) \simeq -\sum_{s=0}^{N} \mathrm{B}^{-1}\,\mathrm{D}^T\,\Lambda[s](t-a)^s \tag{22}$$

If all the matrices in eq.(13) are not constant (time varying problem), the method in [17] cannot be applied easily, because it is very difficult to find the fundamental matrix in step 1 in algorithm (1).

Now, we review the method in [18] and rewrite following algorithm to find the optimal solution of the problem in eq.(12) and eq.(13).

Algorithm 2

1- Apply the DTM about $t=a$ to the state eq.(19) and co-state equations eq.(16), we have an iterative relationship.

2- Apply the DTM about $t=a$ for the $x^*(a)=x_a$, we have $X[0]=x_a$.

3- Let $\lambda^*(a)=A$, where A is unknown constant, then apply the DTM about $t=a$ for the $\lambda^*(a)=A$, we have $\Lambda[0]=A$.

4- Carry out the iterative relationship in step (1) with the $X[0]=x_a$ and $\Lambda[0]=A$, we have $X[s]$ and $\Lambda[s]$ in term of A for all $s=0,1,2,\ldots,N$.

5- Apply the DTM about $t=a$ for the condition $\lambda(b)=\zeta$, we have

$$\sum_{s=0}^{N} \Lambda[s](b-a)^s = \zeta \tag{23}$$

6- To find A, solve the eq.(23).

7- To find $x^*(t)$ and $\lambda^*(t)$, substitute the value of A in the result of step (4), then we have $x^*(t) \simeq \sum_{s=0}^{N} X[s](t-a)^s$ and $\lambda^*(t) \simeq \sum_{s=0}^{N} \Lambda[s](t-a)^s$.

8- By using the result in step (7) and by applying eq.(18), the approximate solution of optimal control is

$$u^*(t) \simeq -\sum_{s=0}^{N} \mathrm{B}^{-1}\,\mathrm{D}^T(t,x^*(t))\Lambda[s](t-a)^s \tag{24}$$

In the Next theorem, we establish an important result that is joint between the DT about two different points for any smooth function $x(t)$.

Theorem 4 Let $x(t):[\mathrm{a},\mathrm{b}]\to \mathbf{R}$ and $\lambda(t):[\mathrm{a},\mathrm{b}]\to \mathbf{R}$ are smooth functions then the relationship between the DT about $t=a$ of $x(t)$ and the DT about $t=b$ of $x(t)$ is given by

$$\chi[s]=\sum_{n=k}^{\infty}\binom{n}{s}\mathrm{X}[n](b-a)^{n-s}, \qquad \forall s=0,1,2,\ldots \tag{25}$$

While the relationship between the DT about $t=b$ of $\lambda(t)$ and the DT about $t=a$ of $\lambda(t)$ is given by

$$\Lambda[s]=\sum_{n=s}^{\infty}(-1)^{n-s}\Upsilon[n]\binom{n}{s}(b-a)^{n-s}, \qquad \forall s=0,1,2,\ldots \tag{26}$$

Proof: Let $\mathrm{X}[s]$ is the DT of $x(t)$ about $t=a$ and $\chi[s]$ is the DT of $x(t)$ about $t=b$, that is

$$x(t)=\sum_{s=0}^{\infty}\mathrm{X}[s](t-a)^{s} \tag{27}$$

and

$$x(t)=\sum_{s=0}^{\infty}\chi[s](t-b)^{s} \tag{28}$$

By using eq.(27) and eq.(28), we have

$$\sum_{s=0}^{\infty}\mathrm{X}[s](t-a)^{s}=\sum_{s=0}^{\infty}\chi[s](t-b)^{s} \tag{29}$$

Set $\eta=t-b$ in eq.(29) we have

$$\sum_{s=0}^{\infty}\mathrm{X}[s](\eta+(\mathrm{b}-a))^{s}=\sum_{s=0}^{\infty}\chi[s]\eta^{s} \tag{30}$$

Now, we rewrite the term $(\eta+(b-a))^{s}$ in eq.(30) by using binomial polynomial

$$\sum_{s=0}^{\infty}\sum_{n=0}^{s}\binom{s}{n}\mathrm{X}[s](b-a)^{s-n}\eta^{n}=\sum_{s=0}^{\infty}\chi[s]\eta^{s} \tag{31}$$

Reorder the double summation in eq.(31), we have

$$\sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\binom{s}{n}\mathrm{X}[s](b-a)^{s-n}\eta^{n}=\sum_{s=0}^{\infty}\chi[s]\eta^{s} \tag{32}$$

By compare the coefficients of η^{s} for all $s=0,1,2,\ldots$ in the eq.(32), one can get

$$\chi[s]=\sum_{n=k}^{\infty}\binom{n}{s}\mathrm{X}[n](b-a)^{n-s}, \qquad \forall s=0,1,2,\ldots \tag{33}$$

Also, let $\Lambda[s]$ is the DT of $\lambda(t)$ about $t=a$ and $\Upsilon[k]$ is the DT of $\lambda(t)$ about $t=b$, that is

$$\lambda(t)=\sum_{s=0}^{\infty}\Lambda[s](t-a)^{s} \tag{34}$$

and

$$\lambda(t)=\sum_{s=0}^{\infty}\Upsilon[s](t-b)^{s} \tag{35}$$

By using eq.(34) and eq.(35), we have

$$\sum_{s=0}^{\infty}\Lambda[s](t-a)^{s}=\sum_{s=0}^{\infty}\Upsilon[s](t-a)^{s} \tag{36}$$

Set $\eta=t-a$ in eq.(36), we have

$$\sum_{s=0}^{\infty}\Lambda[s]\eta^{s}=\sum_{s=0}^{\infty}\Upsilon[s](\eta-(b-a))^{s} \tag{37}$$

Now, we rewrite the term $(\eta-(b-a))^{s}$ in eq.(37) by using binomial polynomial

$$\sum_{s=0}^{\infty}\Lambda[s]\eta^{s}=\sum_{s=0}^{\infty}\sum_{n=0}^{s}(-1)^{s-n}\binom{s}{n}\Upsilon[s](b-a)^{s-n}\eta^{n} \tag{38}$$

Reorder the double summation in eq.(38), we have

$$\sum_{s=0}^{\infty}\Lambda[s]\eta^{s}=\sum_{n=0}^{\infty}\sum_{s=n}^{\infty}(-1)^{s-n}\binom{s}{n}\Upsilon[s](b-a)^{s-n}\eta^{n} \tag{39}$$

By compare the coefficients of η^{s} for all $s=0,1,2,\ldots$ in the eq.(39), one can get

$$\Lambda[s]=\sum_{n=s}^{\infty}(-1)^{n-s}\binom{n}{s}\Upsilon[n](b-a)^{n-s}, \qquad \forall s=0,1,2,\ldots \tag{40}$$

Based on theorem (4), we establish the following algorithm follows:

Algorithm 3

1- Choose N.

2- Apply the DTM about $t=a$ for the $x^{*}(\mathrm{a})=x_{a}$, and the DTM about $t=b$ for the $\lambda^{*}(b)=\zeta$, we have $\mathrm{X}[0]=x_{a}$ and $\Upsilon[0]=\zeta$ respectively .

3- Apply the DTM about $t = a$ to the state equation eq.(19) and the DTM about $t = b$ for co-state equations eq.(16).

4- Apply the result in theorem (4) and substitute the result in step (3), we have a system of equations consist 2N equations with unknown variables $\mathrm{X}[s], \forall s = 1, 2, \ldots, \mathrm{N}$ and $\Upsilon[s], \forall s = 1, 2, \ldots, \mathrm{N}$.

5- Solve the system in step (4) to find the unknown variables $\mathrm{X}[s], \forall s = 1, 2, \ldots, \mathrm{N}$ and $\Upsilon[s], \forall s = 1, 2, \ldots, \mathrm{N}$.

6- To find $x^*(t)$ and $\lambda^*(t)$, use the result in step (5) for $x^*(t) \simeq \sum_{s=0}^{N} X[\mathrm{s}](t-a)^{\mathrm{s}}$ and $\lambda^*(t) \simeq \sum_{s=0}^{N} \Upsilon[s](t-b)^s$.

7- By using the result in step (6) and by applying eq.(18), the approximate solution of optimal control is

$$u^*(t) \simeq -\sum_{s=0}^{N} \mathrm{B}^{-1}\, \mathrm{D}^T(t, x^*(\mathrm{t}))\, \Upsilon[s](t-b)^{\mathrm{s}} \tag{41}$$

4- Illustrated Examples:

This section is devoted some examples to applying the proposed algorithms and showing their ability and accuracy to solve linear or nonlinear QTVOCPs.

Example 1. Consider the problem

$$\min J[u(t), x(t)] = \frac{1}{2}\int_0^1 (u^2(t) + x^2(t))dt \tag{42}$$

subject to

$$\frac{dx(t)}{dt} = u(t) - x(t), \quad x(0) = 1 \tag{43}$$

The exact optimal state space solution of this problem is

$$x^*(t) = \frac{\left(1+\sqrt{2}\right)\mathrm{e}^{-\sqrt{2}(t-2)} + \mathrm{e}^{\sqrt{2}t}\left(\sqrt{2}-1\right)}{\left(1+\sqrt{2}\right)\mathrm{e}^{2\sqrt{2}} + \sqrt{2}-1} \tag{44}$$

and exact optimal control is

$$u^*(t)=\frac{e^{\sqrt{2}t}-e^{-\sqrt{2}(t-2)}}{\sqrt{2}e^{2\sqrt{2}}+e^{2\sqrt{2}}+\sqrt{2}-1} \tag{45}$$

The corresponding TPBVP for this problem is

$$\frac{dx^*(t)}{dt}=-x^*(t)-\lambda^*(\mathrm{t}),\quad x^*(0)=1 \tag{46}$$

$$\frac{d\lambda^*(t)}{dt}=\lambda^*(\mathrm{t})-x^*(t),\quad \lambda^*(1)=0 \tag{47}$$

where the optimal control is

$$u^*(t)=-\lambda^*(t) \tag{48}$$

Application of algorithm (1):
we rewrite eq.(46) and eq.(47) in matrix form as follows:

$$\begin{pmatrix}\dfrac{dx^*(t)}{dt}\\ \dfrac{d\lambda^*(t)}{dt}\end{pmatrix}=\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x^*(t)\\ \lambda^*(t)\end{pmatrix} \tag{49}$$

So, the fundamental matrix for eq.(49) is

$$\Psi(t,0)=e^{\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}t}=\begin{pmatrix}\dfrac{e^{\sqrt{2}t}+e^{-\sqrt{2}t}}{2}+\dfrac{e^{-\sqrt{2}t}-e^{\sqrt{2}t}}{2\sqrt{2}} & -\dfrac{\sqrt{2}(-e^{-\sqrt{2}t}+e^{\sqrt{2}t})}{4}\\ -\dfrac{\sqrt{2}(-e^{-\sqrt{2}t}+e^{\sqrt{2}t})}{4} & \dfrac{e^{\sqrt{2}t}+e^{-\sqrt{2}t}}{2}+\dfrac{e^{\sqrt{2}t}-e^{-\sqrt{2}t}}{2\sqrt{2}}\end{pmatrix} \tag{50}$$

Solve the linear following linear equation for $\lambda(0)$ and $x(1)$

$$\begin{pmatrix}1\\ \lambda(0)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{\sqrt{2}}+e^{-\sqrt{2}}}{2}+\dfrac{e^{-\sqrt{2}}-e^{\sqrt{2}}}{2\sqrt{2}} & -\dfrac{(-e^{-\sqrt{2}}+e^{\sqrt{2}})}{2\sqrt{2}}\\ -\dfrac{(-e^{-\sqrt{2}}+e^{\sqrt{2}})}{2\sqrt{2}} & \dfrac{e^{\sqrt{2}}+e^{-\sqrt{2}}}{2}+\dfrac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2\sqrt{2}}\end{pmatrix}\begin{pmatrix}x(1)\\ 0\end{pmatrix} \tag{51}$$

we find that

$$\lambda(0)=\frac{-e^{\sqrt{2}}+e^{-\sqrt{2}}}{-e^{\sqrt{2}}-\sqrt{2}e^{\sqrt{2}}+e^{-\sqrt{2}}-\sqrt{2}e^{-\sqrt{2}}}=0.3858185960 \tag{52}$$

$$x(1) = -\frac{4e^{-\sqrt{2}}e^{\sqrt{2}}}{-\sqrt{2}e^{\sqrt{2}} - 2e^{\sqrt{2}} + \sqrt{2}e^{-\sqrt{2}} - 2e^{-\sqrt{2}}} = 0.2819695347 \quad (53)$$

Now, we apply the differential transform about $t = 0$ to the system in eq.(49)

$$\mathrm{X}[s+1] = \frac{-\mathrm{X}[s] - \Lambda[s]}{(s+1)}, \quad \forall s = 0,1,2,\ldots,N \quad (54)$$

$$\Lambda[s+1] = \frac{-\Lambda[s] - \mathrm{X}[s]}{(s+1)}, \quad \forall s = 0,1,2,\ldots,N \quad (55)$$

where

$$\mathrm{X}[0] = 1 \quad (56)$$

$$\Lambda[0] = 0.3858185960 \quad (57)$$

For $\mathrm{N} = 10$, we apply the recurrence relation in eq.(54) and eq.(55) , with the values of $\mathrm{X}[0]$ and $\Lambda[0]$, we have

$$\begin{aligned} x_{10}^{*}(t) \simeq{} & 1 - 1.385818596t + 1.000000000t^{2} - 0.4619395320t^{3} + 0.1666666667t^{4} \\ & - 0.04619395320t^{5} + 0.01111111111t^{6} - 0.002199712057t^{7} + 0.0003968253969t^{8} \\ & - 0.00006110311270t^{9} + 0.000008818342153t^{10} - 0.000001110965685t^{11} \end{aligned} \quad (58)$$

$$\begin{aligned} u_{10}^{*}(t) \simeq{} & 0.385818596 + 0.614181404t - 0.385818596t^{2} + 0.204727134t^{3} - 0.0643030993t^{4} \\ & + 0.0204727134t^{5} - 0.00428687328t^{6} + 0.000974891117t^{7} - 0.000153102617t^{8} \\ & + 0.0000270803088t^{9} - 0.00000340228038t^{10} + 4.92369251510^{-7}t^{11} \end{aligned} \quad (59)$$

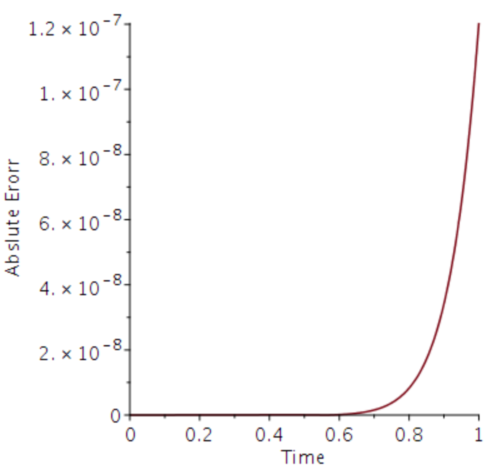


Fig. (1) $\left|x^{*}(t) - x_{10}^{*}(t)\right|$

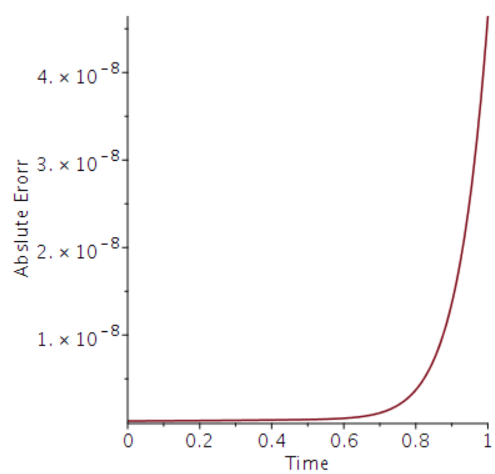


Fig. (2) $\left|u^{*}(t) - u_{10}^{*}(t)\right|$

Application of algorithm (2):

In fact, the algorithm (2) is same the algorithm (1) but in algorithm (2) we will assume the value of $\lambda^*(0)$, say $\lambda^*(0) = A$, instead computed it by using the fundamental matrix. Therefore, we will apply the recurrence relation in eq.(54) and eq.(55) with the values of $X[0]=1$ and $\Lambda[0]=A$. To find the value of A , we use the condition $\lambda(1)=0$, so we have

$$\Lambda[0]+\Lambda[1]+\Lambda[2]+\cdots+\Lambda[N+1]=0 \tag{60}$$

For $N=10$, the approximate optimal state and optimal control are

$$x_{10}^*(t) \simeq 1-\frac{557336}{402171}t+t^2-\frac{557336}{1206513}t^3+\frac{1}{6}t^4-\frac{278668}{6032565}t^5+\frac{1}{90}t^6-\frac{278668}{126683865}t^7 \\ +\frac{1}{2520}t^8-\frac{69667}{1140154785}t^9+\frac{1}{113400}t^{10}-\frac{69667}{62708513175}t^{11} \tag{61}$$

$$u_{10}^*(t) \simeq -\frac{155165}{402171}+\frac{247006}{402171}t-\frac{155165}{402171}t^2+\frac{247006}{1206513}t^3-\frac{155165}{2413026}t^4+\frac{123503}{6032565}t^5-\frac{31033}{7239078}t^6 \\ +\frac{123503}{126683865}t^7-\frac{31033}{202694184}t^8+\frac{123503}{4560619140}t^9-\frac{31033}{9121238280}t^{10}+\frac{123503}{250834052700}t^{11} \tag{62}$$

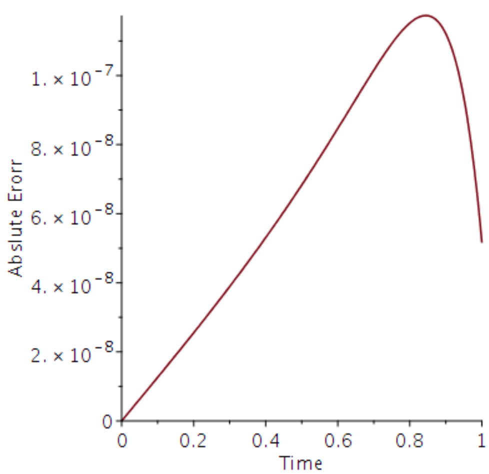


Fig. (3) $\left|x^*(t)-x_{10}^*(t)\right|$

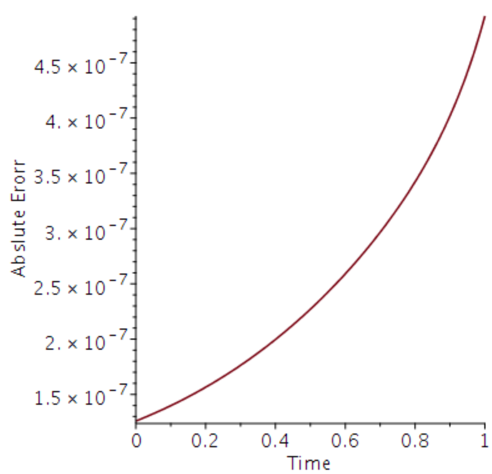


Fig. (4) $\left|u^*(t)-u_{10}^*(t)\right|$

Application of algorithm (3):

In fact, the algorithm (3) is differing from the algorithm (1) and algorithm (2), it is based on applying the DT about different two points. By take the DT about $t=0$ for the eq.(46) and take the DT about $t=1$ for eq.(47), one can get

$$X[s+1]=\frac{-X[s]-\sum_{i=0}^{s}(-1)^{i+s}\binom{s}{i}\Upsilon[s]}{(s+1)},\quad \forall s=1,2,\ldots,N \tag{63}$$

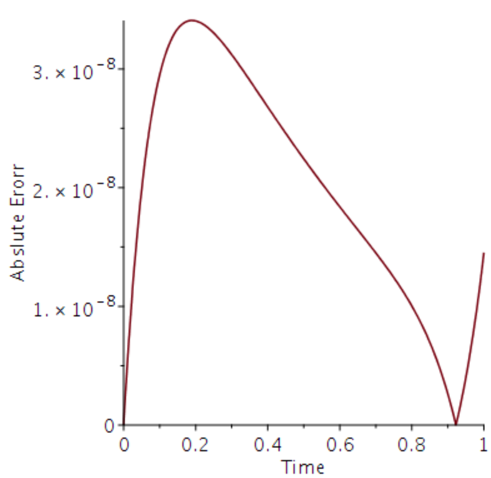
$$\Upsilon[s+1]=\frac{\Upsilon[s]-\sum_{i=0}^{s}\binom{s}{i}X[s]}{(s+1)},\quad \forall s=1,2,\ldots,N \tag{64}$$

where $X[0]=1$ and $\Upsilon[0]=0$.

For $N=10$, the approximate optimal state and control are

$$\begin{aligned} x_{10}^{*}(t)\simeq{}& 1-\frac{25593122995}{18467881009}t+\frac{36935638515}{36935762018}t^{2}-\frac{8530835160}{18467881009}t^{3}+\frac{3077506740}{18467881009}t^{4} \\ &-\frac{9576882}{207504281}t^{5}+\frac{204302595}{18467881009}t^{6}-\frac{39846720}{18467881009}t^{7}+\frac{6833385}{18467881009}t^{8} \\ &-\frac{901015}{18467881009}t^{9}+\frac{180203}{36935762018}t^{10}-\frac{56700}{203146691099}t^{11} \end{aligned} \tag{65}$$

$$\begin{aligned} u_{10}^{*}(t)\simeq{}& \frac{5207384700}{18467881009}(t-1)+\frac{28350}{18467881009}(t-1)^{2}+\frac{1735889400}{18467881009}(t-1)^{3} \\ &+\frac{217350}{18467881009}(t-1)^{4}+\frac{1954260}{207504281}(t-1)^{5}+\frac{411390}{18467881009}(t-1)^{6} \\ &+\frac{8622540}{18467881009}(t-1)^{7}+\frac{454635}{36935762018}(t-1)^{8}+\frac{334015}{18467881009}(t-1)^{9} \\ &+\frac{66803}{36935762018}(t-1)^{10}+\frac{123503}{203146691099}(t-1)^{11} \end{aligned} \tag{66}$$



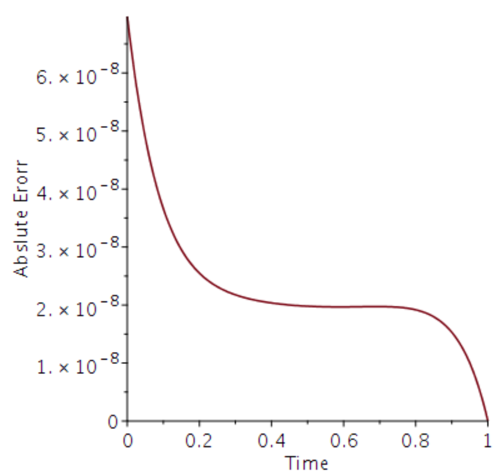


Fig. (5) $\left|x^*(t)-x^*_{10}(t)\right|$ Fig. (6) $\left|u^*(t)-u^*_{10}(t)\right|$

Example 2. Consider the problem

$$\min J[u(t),x(t)]=\frac{1}{2}\int_0^1 (u^2(t)+x^2(t))dt \tag{67}$$

subject to

$$\frac{dx(t)}{dt}=x(t)+t\,u(t),\quad x(0)=1 \tag{68}$$

The exact optimal state space solution of this problem is

$$x^*(t)=\frac{\sqrt{\pi}(\mathrm{erf}(1)-\mathrm{erf(t)})(t+1)\mathrm{e}^{\frac{1}{2}t^2}-2\mathrm{e}^{-\frac{1}{2}t^2}}{\mathrm{erf}(1)\sqrt{\pi}-2} \tag{69}$$

and exact optimal control is

$$u^*(t)=\frac{\mathrm{e}^{\frac{1}{2}t^2}\sqrt{\pi}(\mathrm{erf}(1)-\mathrm{erf(t)})}{\mathrm{erf}(1)\sqrt{\pi}-2} \tag{70}$$

The corresponding TPBVP for this problem is

$$\frac{dx^*(t)}{dt}=x^*(t)-t^2\lambda^*(\mathrm{t}),\quad x^*(0)=1 \tag{71}$$

$$\frac{d\lambda^*(t)}{dt}=-\lambda^*(\mathrm{t})-x^*(t),\quad \lambda^*(1)=0 \tag{72}$$

where the optimal control is

$$u^*(t)=-t\lambda^*(t) \tag{73}$$

Application of algorithm (2):

Assume $\lambda^*(0)=A$ and apply the DT about $t=0$ for eq.(71) and eq.(72), we have

$$(k+1)\mathrm{X}[k+1]=\mathrm{X}[k]-\sum_{s=0}^{k}\Lambda[s]\delta(k-s,2),\quad \forall k=0,1,2,\ldots,N \tag{74}$$

$$(k+1)\Lambda[k+1]=-\Lambda[k]-\mathrm{X}[k],\qquad \forall k=0,1,2,\ldots,N \tag{75}$$

With $\mathrm{X}[0]=1$ and $\Lambda[0]=A$ where A is unknown constant.

For $N=12$, the approximate optimal state and control are

$$x_{12}^{*}(t)\simeq 1+t+\frac{1}{2}t^{2}-\frac{7256611}{8886462}t^{3}+\frac{27842915}{35545848}t^{4}-\frac{24583213}{177729240}t^{5}+\frac{30805069}{355458480}t^{6} -\frac{1101761}{27342960}t^{7}+\frac{67576001}{2843667840}t^{8}-\frac{8244443}{1968693120}t^{9}+\frac{86871557}{51186021120}t^{10} -\frac{27847673}{51186021120}t^{11}+\frac{50330923}{204744084480}t^{12}-\frac{106947461}{2661673098240}t^{13} \quad (76)$$

$$u_{12}^{*}(t)\simeq -\frac{4368844}{1481077}+\frac{5849921}{1481077}t-\frac{2184422}{1481077}t^{2}+\frac{5849921}{8886462}t^{3}-\frac{1092211}{2962154}t^{4} +\frac{40949447}{177729240}t^{5}-\frac{1092211}{17772924}t^{6}+\frac{2507109}{118486160}t^{7}-\frac{1092211}{142183392}t^{8}+\frac{89420221}{25593010560}t^{9} -\frac{1092211}{1421833920}t^{10}+\frac{11471923}{51186021120}t^{11}-\frac{1092211}{17062007040}t^{12}+\frac{12687491}{532334619648}t^{13} \quad (77)$$

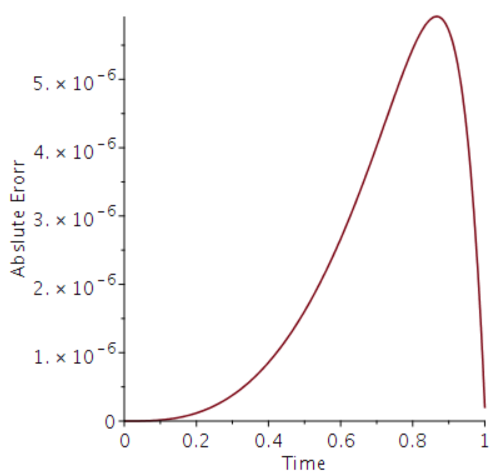


Fig. (7) $\left|x^{*}(t)-x_{12}^{*}(t)\right|$

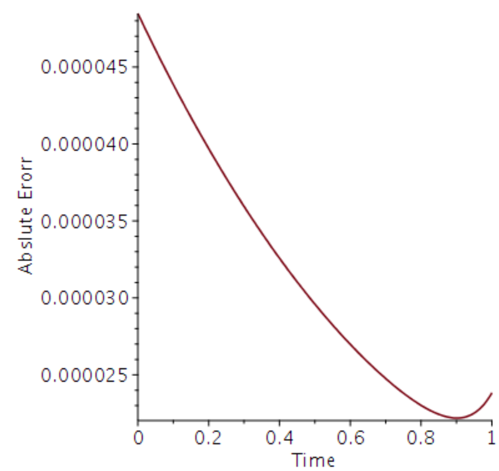


Fig. (8) $\left|u^{*}(t)-u_{12}^{*}(t)\right|$

Application of algorithm (3):

By take the DT about $t=0$ for the eq.(71) and take the DT about $t=1$ for eq.(72), one can get

$$(\mathrm{k}+1)\mathrm{X}[\mathrm{k}+1]=-\mathrm{X}[\mathrm{k}]-\sum_{i=0}^{k}\sum_{s=i}^{N}(-1)^{i-s}\binom{s}{i}\Upsilon[s]\ \delta(\mathrm{k}-\mathrm{i},2),\quad \forall k=0,1,2,\ldots,N \quad (78)$$

$$(k+1)\Upsilon[k+1] = -\Upsilon[k] - \sum_{s=k}^{N} \binom{s}{k} X[s], \quad \forall k = 0,1,2,\ldots,N \tag{79}$$

where $X[0]=1$ and $\Upsilon[0]=0$.

For $N=12$, the approximate optimal state and control are

$$\begin{aligned} x_{12}^{*}(t) \simeq\ & 1+t+\frac{1}{2}t^{2}-\frac{71532527000386}{87597405406795}t^{3}+\frac{137229010978471}{175194810813590}t^{4}-\frac{60563225361896}{437987027033975}t^{5} \\ & +\frac{113692706809922}{1313961081101925}t^{6}-\frac{367925121609268}{9197727567713475}t^{7}+\frac{214325319048457}{9197727567713475}t^{8} \\ & -\frac{33467931181387}{9197727567713475}t^{9}+\frac{22070068166221}{18395455135426950}t^{10}-\frac{87839779462}{437987027033975}t^{11} \\ & +\frac{174180207803}{2627922162203850}t^{12}+\frac{1200190997168}{39856819460091725}t^{13} \end{aligned} \tag{80}$$

$$\begin{aligned} u_{12}^{*}(t) \simeq\ & \frac{22034680171382044}{9197727567713475}(t-1)-\frac{1316735793362}{9197727567713475}(t-1)^{2}+\frac{7339740978428872}{9197727567713475}(t-1)^{3} \\ & +\frac{522629572310674}{1313961081101925}(t-1)^{4}\ \frac{258386859035738}{1313961081101925}(t-1)^{5}+\frac{33054154832257}{437987027033975}(t-1)^{6} \\ & +\frac{56009652412148}{1313961081101925}(t-1)^{7}+\frac{118508286961867}{9197727567713475}(t-1)^{8}+\frac{10688317013961}{3065909189237825}(t-1)^{9} \\ & +\frac{15496254667507}{18395455135426950}(t-1)^{10}+\frac{100569461888}{437987027033975}(t-1)^{11}+\frac{39987793564}{1313961081101925}(t-1)^{12} \\ & +\frac{1256061609}{455506508115334}(t-1)^{13} \end{aligned} \tag{81}$$

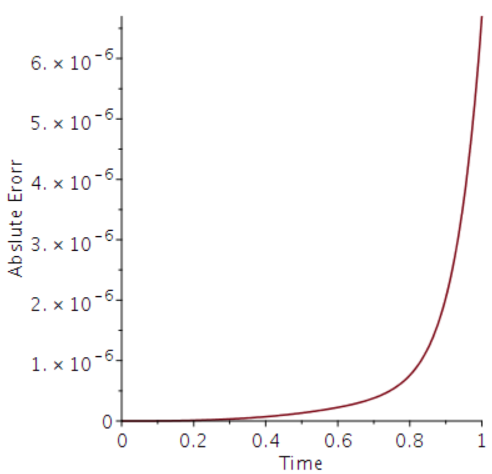


Fig. (9) $\left|x^{*}(t)-x_{12}^{*}(t)\right|$

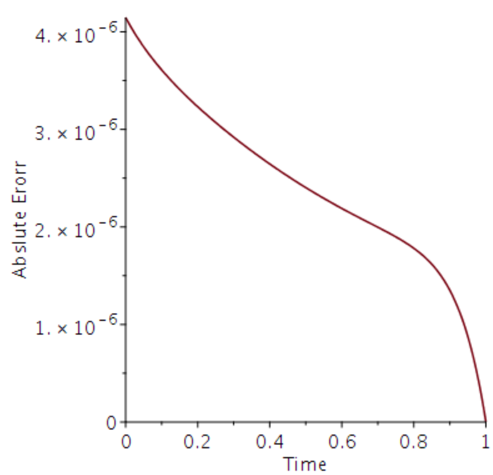


Fig. (10) $\left|u^{*}(t)-u_{12}^{*}(t)\right|$

Example 3. Consider the problem

$$\min J[u(t),x(t)]=\frac{1}{2}\int_0^1 (u^2(t)+(x(t)-1)^2)dt \tag{82}$$

subject to

$$\frac{dx(t)}{dt}=t\,x(t)-u(t), \quad x(0)=1 \tag{83}$$

The exact optimal state space solution and optimal control are very long so we omit it here.

The corresponding TPBVP for this problem is

$$\frac{dx^*(t)}{dt}=t\,x^*(t)-\lambda^*(\mathrm{t}), \quad x^*(0)=1 \tag{84}$$

$$\frac{d\lambda^*(t)}{dt}=1-t\lambda^*(\mathrm{t})-x^*(t), \quad \lambda^*(1)=0 \tag{85}$$

where the optimal control is

$$u^*(t)=\lambda^*(t) \tag{86}$$

Application of algorithm (2):

Assume $\lambda^*(0)=A$ and apply the DT about $t=0$ for eq.(84) and eq.(85), we have

$$(k+1)\mathrm{X}[k+1]=\sum_{s=0}^{k}\mathrm{X}[s]\delta(k-s,1)-\Lambda[k], \quad \forall k=0,1,2,\ldots,N \tag{87}$$

$$(k+1)\Lambda[k+1]=\delta(k,0)-\sum_{s=0}^{k}\Lambda[\mathrm{s}]\ \delta(k-s,1)-\mathrm{X}[k], \qquad \forall k=0,1,2,\ldots,N \tag{88}$$

With $\mathrm{X}[0]=1$ and $\Lambda[0]=A$ where A is unknown constant.

For $N=10$, the approximate optimal state and control are given by

$$x_{10}^*(t)\simeq -\frac{344}{2187}t+\frac{1}{2}t^2-\frac{344}{6561}t^3+\frac{1}{6}t^4-\frac{86}{6561}t^5+\frac{1}{36}t^6-\frac{86}{45927}t^7+\frac{1}{252}t^8-\frac{43}{183708}t^9 +\frac{1}{2520}t^{10}-\frac{43}{2020788}t^{11} \tag{89}$$

$$u_{10}^*(t)\simeq \frac{344}{2187}-\frac{1}{6}t^3+\frac{86}{6561}t^4-\frac{1}{252}t^7+\frac{43}{183708}t^8-\frac{1}{27720}t^{11} \tag{90}$$

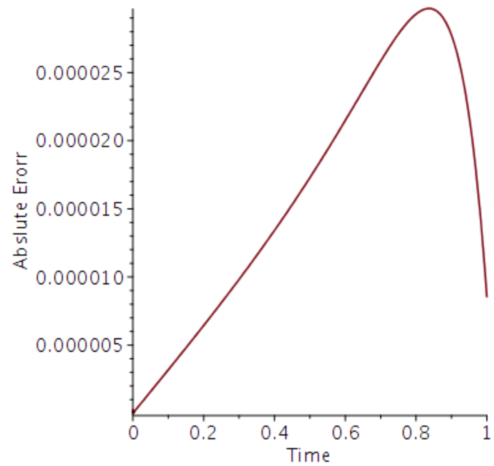


Fig. (11) $\left|x^*(t)-x_{10}^*(t)\right|$

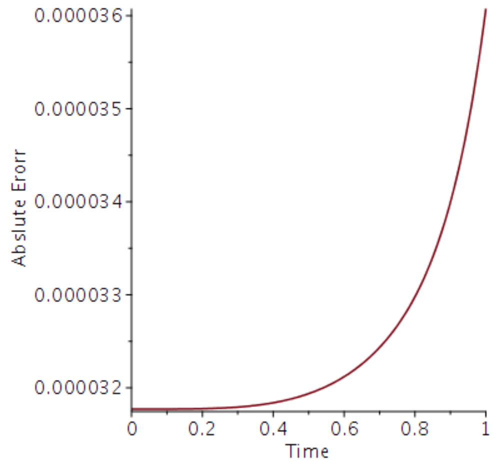


Fig. (12) $\left|u^*(t)-u_{10}^*(t)\right|$

Application of algorithm (3):

By take the DT about $t=0$ for the eq.(84) and take the DT about $t=1$ for eq.(85), one can get

$$(k+1)\mathrm{X}[k+1]=\sum_{s=0}^{k}\mathrm{X}[s]\delta(k-s,1)-\sum_{s=k}^{N}(-1)^{s-k}\binom{s}{k}\Upsilon[s],\quad \forall k=0,1,2,\ldots,N \tag{91}$$

$$(\mathrm{k}+1)\Upsilon[\mathrm{k}+1]=\delta(k,0)-\sum_{s=0}^{k}\Upsilon[\mathrm{s}]\big(\delta(k-s,1)+\delta(k-s,0)\big)-\sum_{s=k}^{N}\binom{s}{k}\mathrm{X}[s],\quad \forall k=0,1,2,\ldots,N \tag{92}$$

where $\mathrm{X}[0]=1$ and $\Upsilon[0]=0$.

For $N=10$, the approximate optimal state and control are

$$\begin{aligned}x_{10}^*(t)\simeq 1&-\frac{2210298118}{14046149655}t+\frac{10922516}{21836721}t^2-\frac{2236870168}{42138448965}t^3+\frac{3670904}{21836721}t^4\\&-\frac{11038626104}{716353632405}t^5+\frac{3325139}{109183605}t^6-\frac{2986530572}{716353632405}t^7+\frac{1182641}{218367210}t^8\\&-\frac{1287587093}{1432707264810}t^9+\frac{1216751}{1965304890}t^{10}-\frac{1255438193}{15759779912910}t^{11}\end{aligned} \tag{93}$$

$$u_{10}^*(t) \simeq -\frac{1357822237009}{286541452962}(t-1) - \frac{358268922881}{716353632405}(t-1)^2 - \frac{176201308793}{716353632405}(t-1)^3 - \frac{9639209737}{79594848045}(t-1)^4 - \frac{61717129582}{716353632405}(t-1)^5 - \frac{26096284811}{716353632405}(t-1)^6 - \frac{1736713211}{143270726481}(t-1)^7 - \frac{2914371254}{716353632405}(t-1)^8 - \frac{115573}{115606170}(t-1)^9 - \frac{490}{21836721}(t-1)^{10} + \frac{8801}{240203931}(t-1)^{11} \quad (94)$$

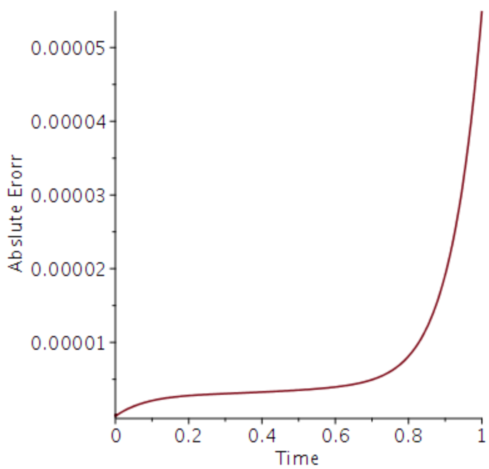


Fig. (13) $\left|x^*(t) - x_{10}^*(t)\right|$

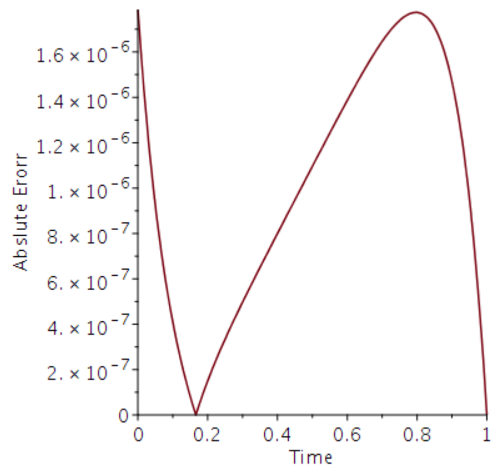


Fig. (14) $\left|u^*(t) - u_{10}^*(t)\right|$

Conclusions

In this paper, we proposed a new algorithm based on applying DTM about the two end point of the time horizon to find an approximate solution for the linear or nonlinear QTVOCPs. This algorithm is better than the algorithm (1) [17], since it doesn't entail computation to the fundamental matrix. Also, the algorithm (1) is not applicable for nonlinear or time varying quadratic optimal control. Furthermore, the proposed algorithm is better than algorithm (2) [18], since the algorithm (2) is based on the assumption for the initial co-state. Three tested examples are used to compare between these algorithms. The absolute errors for the optimal state and optimal control, Fig.(1)-Fig.(14), showed that the proposed algorithm is more reliable , accuracy and efficiency than the other algorithms.

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الحل التقريبي لمسائل السيطرة المثلى التربيعية ذات الوقت المتغير باستخدام طريقة التحويل التفاضلي

سناء لفتة خلف , خلود غركان كاصد

قسم الرياضيات ، كلية العلوم ، جامعة البصرة ، البصرة ، العراق

المستخلص

نظراً لعدم وجود طريقة تحليلية عامة لحل مسائل الامثلية التربيعية ذات الوقت المتغير، فقد لجا العديد من الباحثين الى استخدام الطرق العددية لهذا الغرض. وهذا حفز بعض الباحثين لأستخدام طريقة التحويل التفاضلي لإيجاد حل تقريبي لمسائل التحكم الأمثل التربيعية الخطية وحالات خاصة من غير الخطية. في هذه البحث ، اقترحنا خوارزمية جديدة لإيجاد حل تقريبي لمسائل السيطرة التربيعية المثلى ذات الوقت المتغير بحالتيها الخطية وغير الخطية .ولإظهار موثوقية ودقة وكفاءة الخوارزمية المقترحة ، تم تقديم بعض الأمثلة التوضيحية.