# Bifurcation Solutions of Fourth Order Non-linear Differential Equation Using a Local Method of Lyapunov –Schmidt

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## Abstract

In this paper, the bifurcation solutions of the boundary condition problem has been studied by using the local method from Lyapunov-Schmidt. We reduce the bifurcation equation and make it in the form of Operator equation. In addition, the finite dimensional reduction theorem for the bifurcation equation is given by the nonlinear system of fourth order equations. We investigate the analysis system of the bifurcation equation, we also find the Discriminant set of corresponding to the nonlinear differential equation by using (Maple 2016) program. The classification of the equilibrium points of the Dynamical System are discussed. And the phase portrait of boundary condition problem is found in three dimensional.

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## 1. Introduction

There are many examples of bifurcation we find in in some sciences such as Mathematics, Engineering , Mechanics and Physics may be written as [1],

$G(x,\lambda) = 0, x \in O \subset E,\ \ x \in X\ \ ,\lambda \in R^n.$

Where G is a nonlinear smooth Fredholm map of index zero, E and F are Banach spaces and $O$ an open subset in E. For these problems, we are working on a method of reduction the equation (1) to finite dimensional equation,

$$\Phi(\tau,\lambda) = 0\ , \tau \in M \qquad \dots (1)$$

M is asmooth finite-dimensional in submanifold $O$, The reduction method of finite-dimensional introduced by Lyapunov [2], Schmidt [3],though this method, solution of equation(1). Vainberg [4], Sapronov[5,6] and Loginv [7] used the local method of Lyapanov – Schmidt on the conditions that equation(2) has all analytical and the topological properties of equation (1) (bifurcation diagram ,multiplicity, etc). Danilova [8], Gnezdilov [9], Darinskii [10], Shveriova [11], Abdul Hussain [12,13], Mohammed [14], and others they studied the theory of Fredholm functionals on Banach manifolds and its applications. Dariski [15] used the method of Lyapunov-Schmidt reduced to find bifurcation solutions of the problem,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0,$$

With boundary condition, $u(0) = u(\pi) = u''(0) = u''(\pi) = 0,\ \ x \in [0,\pi]$, Abdul Hussain [12] has been studied the equation,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^2 = 0,$$

With boundary condition, $u(0) = u(\pi) = u''(0) = u''(\pi) = 0,\ \ x_1, x_2 \in [0,1]$,

Abdul Hussain [13] studied the following equation,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = \psi,$$

With boundary condition, $u(0) = u(\pi) = u''(0) = u''(\pi) = 0$.

They demonstrated that the nonlinear system of two algebraic equations gives the bifurcation equation matching to the elastic beams equation.

Shanan [16] studied the following equation, in three dimension,

$$\delta\frac{d^4z}{dx^4} + \alpha\frac{d^2z}{dx^2} + \beta z + z^2 + z^3 = \psi,$$

With boundary condition, $z(0) = z(\pi) = z''(0) = z''(\pi) = 0$.

showed the bifurcation equation corresponding to above problem.

Resan [17] studied by used local method of Lyapunov-Schmidt reduction the following equation,

$$y'''' + \alpha y'' + \beta y + yy'' + (y')^2 + y^2 + y^3 = 0$$

Bifurcation solutions for the fourth-order nonlinear differential equation, classify the equilibrium points and the phase portrait in two dimension.

Qaasim [18] studied by using local Lyapunov-Schmidt reduction the following equation,

$$\lambda_1 w'''' + \lambda_2 w'' + \lambda_3 (ww'' + (w')^2) + 3(w^2 w'' + 2w(w')^2) = 0$$

showed the bifurcation equation corresponding to above problem.

Sadiq [19] studied the imperfect bifurcation of traveling wave solutions by used Lyapunov-Schmidt reduction the following equation,

$$u_t - u_{tyy} + 3uu_y + \big(2k + \epsilon_1 (y - ct)\big)u_y + \epsilon_2 u_{yy} = 2u_y u_{yy} + uu_{yyy}$$

showed the bifurcation equation corresponding to above problem and classify the equilibrium points and the phase portrait in two dimension.

Jassim [20] studied by used local method of Lyapunov-Schmidt reduction the following equation,

$$\frac{d^4u}{dx^4} + \alpha \frac{d^2u}{dx^2} + u + u^3 + u^5 = 0,$$

With the boundary conditions $u(0) = u(1) = u''(0) = u''(1)$, showed the bifurcation equation corresponding to above problem and classify the equilibrium points and the phase portrait in one dimension. In this paper we used the local method of Lyapunov-Schmidt to find the bifurcation solutions of non-linear differential equations of the fourth order in three parameters,

$$\alpha \frac{d^4u}{dx^4} + \beta \frac{d^2u}{dx^2} + \sigma u + u\frac{du}{dx} = 0 \qquad \dots (2)$$

Equation (2) has been studied with the boundary conditions,

$$u(0) = u(\pi) = u''(0) = u''(\pi) \qquad \dots (3)$$

We shall introduced basic definition.

**Definition 1.1** [21]

Suppose that( E, M are Banach spaces) and a linear continuous operator A: E → M. The operator A is called Fredholm operator, if the kernel of A,( ker(A)) and the Cokernel of A,( $Coker(A)$) is finite dimensional.The Rang of A ( Im(A)), is closed in M. The number dim (Ker A ) - dim (Coker A) is called Fredholm index of the operator A.

**Theorem.1.1** [22]

Suppose E and M are two real Banach spaces and $F(u,\lambda)$ is a $C^1$ continuous map in a neighborhood of a point $(u_0,\lambda_0)$ with a range in M such that $F(u_0,\lambda_0) = 0$ and the Fredholm operator $F_x(u_0,\lambda_0)$ is a linear Fredholm operator. Then each solutions (u, λ) of $F(u,\lambda) = 0$ near $(u_0,\lambda_0)$ (with λ fixed) are in one-to-one correspondence with the solutions of a finite dimensional system of $N_1$ real variables and $N_0$ equations. Such that,

$$N_0 = \dim Ker(L) \; and \; N_1 = \dim Coker(L)\,, \left(L = F_x(u_0,\lambda_0)\right).$$

**2. Reduction to Bifurcation Equation**

An efficient method for studying equation (2) with the boundary conditions (3) is the Lyapunov-Schmidt method. By reducing equation (2) we will make it in the from of operator equation, that is,

$$f(u,\lambda) = \alpha\frac{d^4u}{dx^4} + \beta\frac{d^2u}{dx^2} + \sigma u + u\frac{du}{dx} \qquad \dots (4)$$

where $F: E \to M$ is nonlinear Fredholm operator of index zero form Banach space $E$ to Banach space $M$, $E = C^4([0,\pi],R)$ is the space of all continuous function that have derivative of order at most four, $M = C^0([0,\pi],R)$, $x \in [0,\pi]$. Is the space of all continuous functions and $u = u(x)$, $x \in [0,\pi]$,$\lambda = (\alpha,\beta,\sigma)$. the bifurcation solutions of equation (2) are corresponding to the bifurcation equation ,

$$f(u,\lambda) = 0 \qquad \dots (5)$$

Now, we will prove the finite dimensional reduction theorem for the equation (5)

**Theorem 2.1.**

The bifurcation equation corresponding to the equation (5) is given by the following nonlinear system of three equations,

$$\begin{pmatrix} q_1x_1 - A_1x_1x_2 - A_2x_2x_3 \\ q_2x_2 + B_1x_1^2 - B_2x_1x_3 \\ q_3x_3 + x_1x_2 \end{pmatrix} + o(|\tau|^3) + O(|\tau|^3)O(\gamma) = 0 \qquad \dots (6)$$

where, $\tau = (x_1, x_2, x_3)$, $\lambda = (q_1, q_2, q_3) \in R^3$, $A_1 = \frac{1}{\sqrt{2\pi}}$, $A_2 = \sqrt{\frac{2}{\pi}}$, $B_1 = \frac{1}{\sqrt{2\pi}}$, $B_2 = \sqrt{\frac{2}{\pi}}$, $C = \frac{5}{\sqrt{2\pi}}$,. Furthermore, the solutions of equation (5) are one-to-one correspondence with the solutions of the nonlinear.

**Proof:** Let $f(u, \lambda) = 0$

Now, the Frèchet derivative of nonlinear operator $f(u, \lambda)$ at the point $(0, \lambda)$ is;

$$f(u + th, \lambda) = \alpha \frac{d^4(u + th)}{dx^4} + \beta \frac{d^2(u + th)}{dx^2} + \delta(u + th) + (u + th)\frac{d(u + th)}{dx}$$

By differentiate $f$ with recpect to $t$ and let t = 0, we get

$$\frac{\partial}{\partial t} f(0 + th, \lambda) = \alpha \frac{d^4 h}{dx^4} + \beta \frac{d^2 h}{dx^2} + \sigma h \qquad \ldots (7)$$

And hence the linearized equation corresponding to the equation(2) is given by, $\check{A}h = 0$ ; $h \epsilon E$

$$\check{A} = \frac{\partial}{\partial t} f(0, \lambda) = \alpha \frac{d^4}{dx^4} + \beta \frac{d^2}{dx^2} + \sigma, \; x \in [0, \pi] \, , h(0) = h(\pi) = h''(0) = h''(\pi) = 0$$

The solution of linearized equation which satisfied the boundary conditions is given by $e_{Ƥ} = c_{Ƥ} \sin(Ƥx)$, $Ƥ = 1,2,3, \ldots$,the characteristic equation corresponding to this solution is,

$\alpha m^4 + \beta m^2 + \sigma = 0$

This equation gives in the space of parameters $(\alpha, \beta, \sigma)$ characteristic planes $l_{Ƥ}$. The characteristic planes $l_{Ƥ}$ consist of the points $(\alpha, \beta, \sigma)$ for which the linearized equation has non-zero solutions. The point of intersection of characteristic planes in the space of parameters $(\alpha, \beta, \sigma)$is a bifurcation point. lead to bifurcation along the three modes

$$e_1 = c_1 \sin(x), \;\; e_2 = c_2 \, sin(2x), \;\; e_3 = c_3 \, sin(3x)$$

Where, $c_1 = c_2 = c_3 = \sqrt{\frac{2}{\pi}},$

This means that $e_1, e_2$ *and* $e_3$ are orthonormal basis in the null space ker(A). Let $N = ker(A) = span\{e_1, e_2, e_3\}$, then the space $E$ can be decomposed in direct sum of two subspaces, $N$ and the orthogonal complement of $N$,

$$E = N \oplus N^{\perp}, \qquad N^{\perp} = \left\{ ž \in E: \int_0^{\pi} že_{Ƥ} \, dx = 0, Ƥ = 1,2,3 \right\}$$

Similarly, the space $M$ can be represented in the direct sum of two subspaces $\breve{N}$, and the orthogonal complement to $\breve{N}$. $M = N \oplus \breve{N}^{\perp}$ , $\breve{N}^{\perp} = \{ ώ \in M: \int_0^{\pi} ώe_{Ƥ} \, dx = 0, \; wherever, Ƥ =$

$1,2,3\}$. There exist two projections $\text{þ}: E \to N$ also, $(\mathrm{I} - \text{þ}): E \to N^{\perp}$ ($I$ is the identity operator),such that, $\text{þ}\, u = v$ and $(\mathrm{I} - \text{þ})\, u = \breve{Z}$

And hence every vector $u \in E$ can be written in the unique form, $u = v + \breve{Z}$

$$v = \sum_{i=1}^{3} x_i e_i \in N, \breve{Z} \in N^{\perp}, x_i = \langle u, e_i \rangle$$

similarly, there are exist two projections

$\text{Q}: M \to \breve{N}$ and $(I - \text{Q}): M \to \breve{N}^{\perp}$

such that $F(u,\lambda) = F_1(u,\lambda) + F_2(u,\lambda) = \text{Q}F(u,\lambda) + (I - \text{Q})F(u,\lambda)$,

$$F_1(u,\lambda) = \text{Q}F(u,\lambda) = \sum_{i=1}^{3} v_i\,(u,\lambda) e_i \in \breve{N},$$

$$F_2(u,\lambda) = (I - \text{Q})F(u,\lambda) \in \breve{N}^{\perp}, \quad when, \quad v_i(u,\lambda) = \langle F(u,\lambda), e_i \rangle$$

Accordingly, equation (5) can be written in the form, $QF(u,\lambda) = 0, (I - Q)F(u,\lambda) = 0$,

or $\text{Q}F\left(v + \breve{Z},\ \lambda\right) = 0, \quad (I - \text{Q})F\left(v + \breve{Z},\ \lambda\right) = 0$,

By using implicit function theorem, there exists a smooth map $\Theta: N \to N^{\perp}$ (depending on $\lambda$) such that $\Theta\,(u,\lambda) = \breve{Z}$ and, $(\mathrm{I} - \text{Q})\; F\;(v + \Theta(v,\lambda),\lambda) = 0$.

To find the solution for $F(u,\lambda)$ =0 in the neighbourhood of the point $\breve{U} = 0$, it is sufficient to find the solutions of the equation

$$\text{Q}F(v + \Theta(v,\lambda),\lambda) = 0 \qquad \dots (8)$$

This equation is called bifurcation equation for equation (5). This means, the bifurcation quation is $\Phi(\tau,\lambda) = 0$, $\tau = (x_1, x_2, x_3)$, $\lambda = (\alpha,\beta,\sigma)$.

Where $\Phi(\tau,\lambda) = F_1(v + \Theta(v,\lambda),\lambda)$,

Equation (5) can be written in the form, $F\left(v + \breve{Z},\lambda\right) = A\left(v + \breve{Z}\right) + K(v + \breve{Z})$

Simple calculations show that, $K\left(v + \breve{Z}\right) = Kv$ +terms consists elements of $\breve{Z}$

Hence, $F\left(v + \breve{Z},\lambda\right) = Av + uv' + \cdots$ Where the dots denote the terms that consist of the element $\breve{Z}$ . Hence,

$$\Phi\,(\tau,\lambda) = \text{Q}F\left(v + \breve{Z},\lambda\right) = \sum_{i=1}^{3} \langle Av + uv', e_i \rangle = 0 \qquad \dots (9)$$

By using the properties of the inner product $\langle .,. \rangle_H$ in Hilbert space $L_2([0,1],R)$ and by some calculations of equation (9) , implies that,

$$\langle Av + uv', e_1 \rangle e_1 + \langle Av + uv', e_2 \rangle e_2 + \langle A\,v + uv', e_3 \rangle e_3 = 0 \qquad \dots (10)$$

By using the properties of the inner product we have,

$$\langle Av + uv', e_1 \rangle = \langle Av , e_1 \rangle + \langle uv', e_1 \rangle \quad \text{... (11)}$$

Where, $\text{ɥ}_1 = Ae_1 = \vartheta_1(\lambda)e_1$, $\text{ɥ}_2 = Ae_2 = \vartheta_2(\lambda)e_2$, $\text{ɥ}_3 = Ae_3 = \vartheta_3(\lambda)e_3$

Substituting these results in (10) lead to,

$$\Phi(\tau,\lambda) = \begin{pmatrix} \text{ɥ}_1 x_1 - A_1 x_1 x_2 - A_2 x_2 x_3 \\ \text{ɥ}_2 x_2 + B_1 x_1^2 - B_2 x_1 x_3 \\ \text{ɥ}_3 x_3 + C x_1 x_2 \end{pmatrix} + o(|\tau|^3) + O(|\tau|^3)O(\gamma) = 0 \quad \text{... (12)}$$

$$A_1 = \frac{1}{\sqrt{2\pi}}, \quad A_2 = \sqrt{\frac{2}{\pi}}, B_1 = \frac{1}{\sqrt{2\pi}}, B_2 = \sqrt{\frac{2}{\pi}}, \quad C = \frac{5}{\sqrt{2\pi}},$$

## 3. Analysis system of Bifurcation

Theorem (1.1) tell us that there is a one-to-one function corresponding between the solutions of equation (5) and the solutions of the equation $\Phi(\tau,\lambda) = 0$.

The equation $\Phi(\tau,\lambda) = 0$, is equivalent in the neighborhood of point zero to the equation,

$$\Phi_0(\tau,\lambda) = 0 \quad \text{... (13)}$$

$$\Phi_0(\tau,\lambda) = \begin{pmatrix} \text{ɥ}_1 x_1 - \frac{x_1 x_2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}} x_2 x_3 \\ \text{ɥ}_2 x_2 + \frac{x_1^2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}} x_1 x_3 \\ \text{ɥ}_3 x_3 + \frac{5 x_1 x_2}{\sqrt{2\pi}} \end{pmatrix} \quad \text{... (14)}$$

where $\text{ɥ}_1, \text{ɥ}_2 \text{ɥ}_3, \in R$.

This means the study of the bifurcation set of the equation $\Phi(\tau,\lambda) = 0$ is equivalent to the study bifurcation set of equation (13). The solutions of the equation (13) are degenerate on the surface given by the equation $|J| = 0$, where $J$ is the Jacobean matrix from the system (14), so we used a program(Mathematics 11). The discriminant of the bifurcation set of equation (13) in three dimention is give by following surfaces, (in figure.1) $S_1.S_2.S_3 = 0$. where,

$$\left.\begin{aligned} S_1 &= q_1q_2q_3 \\ S_2 &= \left(-40q_1q_2q_3 + q_2q_3^2 + \sqrt{q_2^2q_3^3(-40q_1+q_3)}\right)^2 \\ S_3 &= \left(40q_1q_2q_3 + q_2q_3^2 + \sqrt{q_2^2q_3^3(-40q_1+q_3)}\right)^2 \end{aligned}\right\} \quad \ldots (15)$$

To find the Discriminant set of equation (13) so to we used a program (Maple 2016) we found that the discriminant of the bifurcation set in 2D of equation (13) in two dimention can be found by substituting each solution in the following equation in Fig.1.

$S_1.S_2.S_3 = 0$, we will install a value $q_3$ and then find every part of the Discriminant set in $the\ q_1q_2 - plans$, in( figure.2).

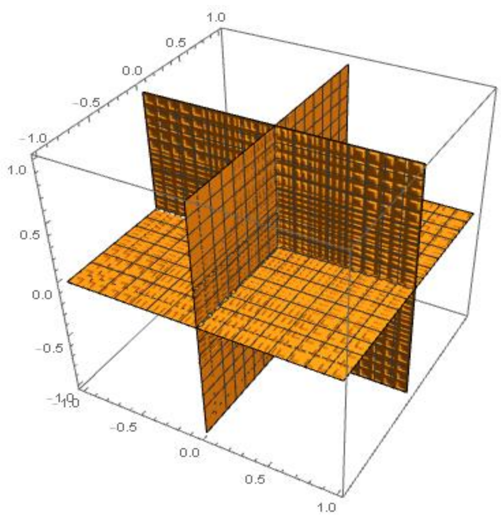



Fig. 1: Describe the Discriminant set in 3D of equation(11)

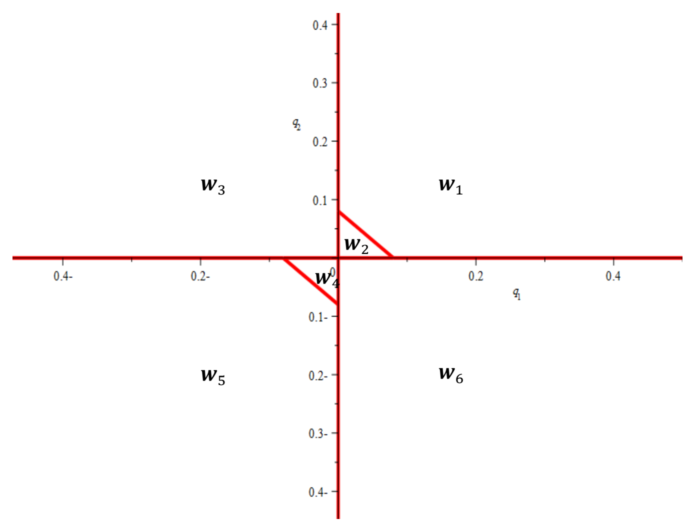



Fig. 2: Describe the Discriminant set in 2D of equation(11) by install a value $q_3$

In $the\ q_1q_2 - plans$ when $q_3 = 4$, The bifurcation set of equation (13) partitioned the plan of parameters into fifteen disjoint sets every set has a fixed number of solutions of equation (13). In the sets $W_1$, $W_2$ and $W_6$ there are one solutions with topological indices 1. In the set $W_3$ there are three solutions with topological indices1,-1,1. In the set $W_4$ there are three solutions with topological indices 1,-1,-1. . In the set $W_5$ there are five solutions with topological indices -1,-1,1,1,1.

## 4. Application for dynamical system

We noticed that the solutions of equation (13). Can be consider the equilibrium points of the dynamical system [23],

$$\begin{cases} \dot{x}_1 = q_1 x_1 - \frac{x_1 x_2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}} x_2 x_3 \\ \dot{x}_2 = q_2 x_2 + \frac{x_1^2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}} x_1 x_3 \\ \dot{x}_3 = q_3 x_3 + \frac{5 x_1 x_2}{\sqrt{2\pi}} \end{cases} \quad ...(16)$$

We can classify the (equilibrium points) of system (16). In the set $\boldsymbol{W_1}$ we choose $q_1 = 3.801$, $q_2 = 8.088$ and $q_3 = 4$, for these values we have the following one real solution (equilibrium point) of system (16). $P_1 = (0,0,0)$, After testing the eigenvalues of the jacobian matrix of system (16) [4] at point we found that the point is Node stable. In the set $\boldsymbol{W_2}$ we choose $q_1 = 0.145$, $q_2 = 0.063$ and $q_3 = 4$, for these values we have the following one real solution (equilibrium point) of system (16), $P_1 = (0,0,0)$. After testing the eigenvalues of the jacobian matrix of system (16) at point we found that the point $P_1$ is Node unstable. In the set $\boldsymbol{W_3}$ we choose $q_1 = -0.192$ , $q_2 = 0.745$ and $q_3 = 4$, for these values we have the following three real solution (equilibrium point) of system (16), $P_1 = (0,0,0)$

$P_2 = (-1.0138262164618708, -0.355339999736443, -0.17965019541209865)$

$P_3 = (1.0138262164618674, -0.35533999973644215, 0.1796501954120981)$

After testing the eigenvalues of the jacobian matrix of system (16) at all point we found that the point $P_1$ is Node unstable and $P_2, P_3$ are Focus unstable .In the set $\boldsymbol{W_4}$ we choose $q_1 = -0.064$, $q_2 = -0.095$, and $q_3 = 4$, for these values we have the following one real solution (equilibrium point) of system (13), $P_1 = (0,0,0)$

$P_2 = (0.311782, 1.23812, -0.192502), P_3 = (-0.311782, 1.23812, 0.192502)$

After testing the eigenvalues of the jacobian matrix of system (16) at all point we found that the point $P_1$ Node stable and $P_2, P_3$ are saddle unstable. In the set $\boldsymbol{W_5}$ we choose $q_1 = -4.015$ , $q_2 = -15.296$ and $q_3 = 4$, for these values we have the following one real solution (equilibrium point) of system (16), $P_1 = (0,0,0)$

$$P_2 = (5.502413245751388, 3.717242789371301, -10.19985999166625)$$

$$P_3 = (-5.502413245751395, 3.717242789371305, 10.199859991666253)$$

$$P_4 = 7.807586722777821, -2.7145914795189023, 10.56918202020681$$

$$P_5 = -7.8075867227778355, -2.7145914795189063, -10.569182020206815$$

After testing the eigenvalues of the jacobian matrix of system (16) at all point we found that the point $P_1$ Node is stable and $P_2, P_3$ are saddle unstable and $P_4, P_5$ are focus unstable. In the set $\boldsymbol{W_6}$ we choose $q_3 = 6.212$ , $q_3 = -16.62$ and $q_3 = 4$, for these values we have the following one real solution (equilibrium point) of system (16), $P_1 = (0, 0, 0)$ After testing the eigenvalues of the jacobian matrix of system (16) at all point we found that the point $P_1$ saddle unstable. The phase portrait of system (16) in each set $W_i$, ($i$=1,…,6) is given below. All figures has been drawn Phase Patriot by program ( Mathematica 11).

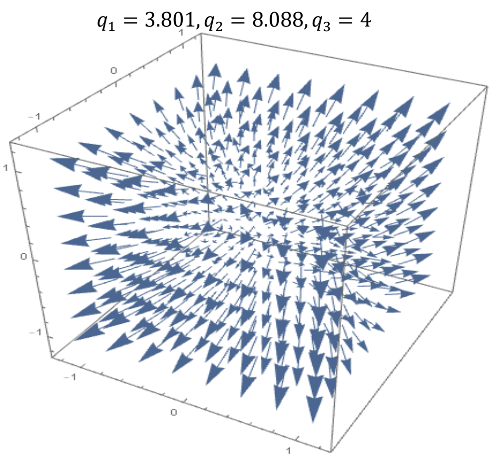
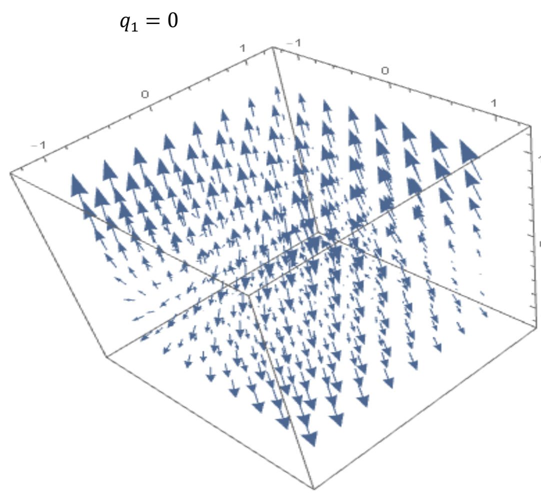
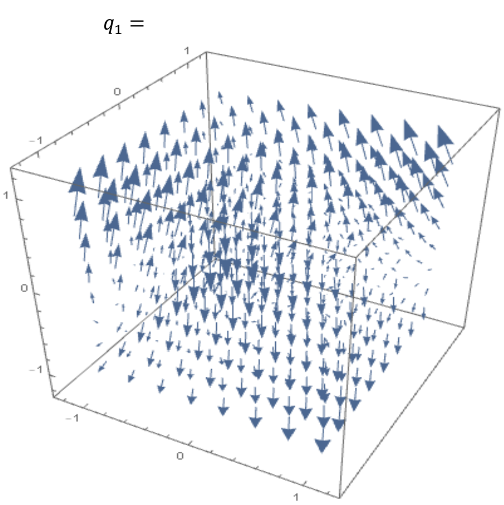
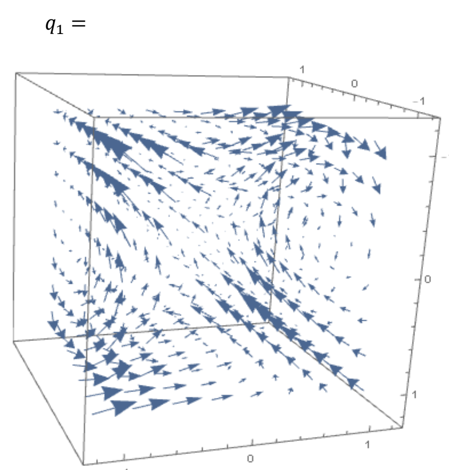
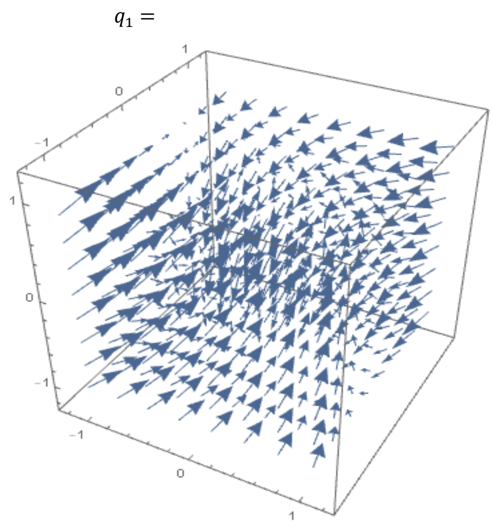


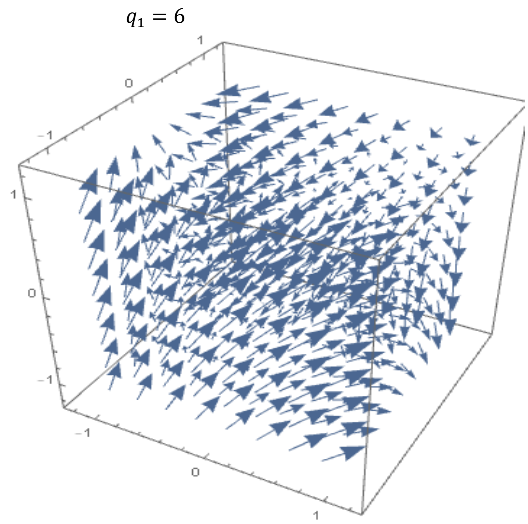

Fig. 3. The phase portrait for set W1






**Conclusions**

In this paper, we using the Lyapunov-Schmidt method locals to find bifurcation solutions of the in nonlinear equation. In this work we explain the search for solutions for the bifurcation of these boundary value problem (1), and we explained that the bifurcation equation corresponding to the problem (1), is calculated using nonlinear equations. We find the theoretical of the discriminative collection of the problem and get an engineering description of it (1), and we also have other outcomes. By means of the engineering drawings, there are six areas and all area has the number of solutions. There is a real solution in regions $w_1$, $w_2, w_3, w_4, w_5$ and $w_6$, where there are only one real solutions for regions $w_1$, $w_2$ and $w_6$ and where there are three real solutions for regions $w_3, w_4$, and where there are five real solutions for regions $w_5$. And in the $w_1$ region there are one balance points (node stable) which means that there are one real solutions, And in the $w_2$ region there are one balance points (node unstable) which means that there are one real solutions, And in the $w_3$ region there are two balance points (node unstable and focus unstable ) which means that there are two real solutions , And in the $w_4, w_5$ region there are two balance points (node stable and saddle unstable) which means that there are two real solutions, And in the $w_6$ region there are one balance points (saddle unstable) which means that there are one real solutions.

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## حلول التفرع لمعادلة تفاضلية من الرتبة الرابعة باستخدام طريقة ليبانوف-شمدت المحلية

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**المستخلص**

في هذا البحث درسنا بأستخدام طريقة ليبانوف-شمدت المحلية( حلول التفرع للمعادلة التفاضلية اللاخطية من الرتبة الرابعة). اثبتنا ان معادلة التفرع ( المجموعة المميزة) المقابلة لهذه المعادلة تعطى على شكل نظام لاخطي من ثلاث معادلات جبرية. كذلك وجدنا المعادلة المميزة لهذه المعادلة مع رسمها بأستخدام برنامج (MAPLE) مع وصف هندسي لمخطط التفرع. وقمنا بتصنيف نقاط الاتزان (equilibrium points) في ثلاثة ابعاد, ثم رسمنا صورة المرحله للنظام الديناميكي ( Phase Portrait) في ثلاث ابعاد.